A variational approach to uniqueness of ground states for certain quasilinear PDEs

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Problem

Given two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to [0, \infty)$, and an integer $p > 1$, consider a system with energy

$$E(u) = \frac{1}{p} \int_{\mathbb{R}^n} |\nabla u|^p + \int_{\mathbb{R}^n} f(u),$$

where u belongs to the space of states

$$S = \left\{ u : \mathbb{R}^n \to \mathbb{R}, \ \int_{\mathbb{R}^n} g(u) = 1 \right\}.$$

Assumptions: f and g are even and nondecreasing (or in general f is the difference of two monotone functions).

Questions

- *Question 1*: What is the lowest possible energy of the system, and what is a corresponding state? i.e.,

  $$E_0 := \inf \{E(u),\ u \in S\}? \qquad u_0 = \text{Argmin}\{E(u),\ u \in S\}?$$

  E_0, the lowest possible energy is called *the ground state energy*, and an associated lowest energy state, $u_0 \in S$ with $E(u_0) = E_0$, is called *a ground state*.

- *Question 2:* Is the ground state unique, or are they degenerate (i.e., does the system have more than one ground states)? This is a question on uniqueness of the minimizers to the variational problem

  $$(P): \quad \inf \left\{ E(u) := \frac{1}{p} \int_{\mathbb{R}^n} |\nabla u|^p + \int_{\mathbb{R}^n} f(u),\ \int_{\mathbb{R}^n} g(u) = 1 \right\}$$

Motivation

Answers to these questions allow to derive the sharp constants and the optimal functions of many geometric inequalities which are useful to study stability properties for some evolutive PDEs, such as

- the heat equation:

$$\partial u/\partial t = \Delta u$$

- the porous medium equation

$$\partial u/\partial t = \Delta u^m, \ m > 1$$

- the Yamabé flow: For $2^* := \frac{2n}{n-2}$

$$\frac{\partial u^{2^*-1}}{\partial t} = \frac{n-2}{4(n-1)}\Delta u + s(t) u^{2^*-1}; \ s(t) \text{ is s.t. } \int_{\mathbb{R}^n} |u|^{2^*} = 1$$

Motivation (continued)

- Answer to *Question 1* enables to derive explicitly the sharp constant and AN optimal function of some geometric inequalities, such as
  - Sobolev inequalities: $f(u) = 0$ and $g(u) = |u|^{p^*}$, where $p^* = np/(n-p)$ and $n > p$; $\quad \forall u \in C_c^1(\mathbb{R}^n)$,

    $$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C_{sharp}(n,p)\|\nabla u\|_{L^p(\mathbb{R}^n)}$$

    where $\|u\|_{L^r(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |u|^r\right)^{1/r}.$
  - Gagliardo-Nirenberg inequalities: $f(u) = |u|^q/q$ and $g(u) = |u|^s$, where $1 < q < s < p^*$ and $n > p$;

    $$\|u\|_{L^s(\mathbb{R}^n)} \leq C_{sharp}(n,p,q,s)\|\nabla u\|^{\theta}_{L^p(\mathbb{R}^n)}\|u\|^{1-\theta}_{L^q(\mathbb{R}^n)}$$

Motivation (continued)

- Logarithmic Sobolev inequality: $f(u) = -|u| \ln(|u|)$, $g(u) = |u|$, and $p = 2$;

$$\int_{\mathbb{R}^n} v^2 \ln(v^2) \mathrm{d}\mu(x) \leq 2 \int_{\mathbb{R}^n} |\nabla v|^2 \mathrm{d}\mu(x)$$

where

$$u := v^2 u_0; \; u_0(x) = (2\pi)^{-n/2} \mathrm{exp}(-|x|^2/2); \; \mathrm{d}\mu(x) = u_0(x)\mathrm{d}x.$$

In fact, an optimal function in these inequalities is a ground state, u_0, of the corresponding system, and the sharp constant, C_{sharp}, is obtained explicitly in terms of the ground state energy, $E_0 = E(u_0)$.

• Answer to *Question 2* allows to identify ALL the optimal functions of these geometric inequalities.

Illustration: A baby example

Toy model evolution equation: ODE gradient flow

$$\dot{x}(t) = -\nabla F\left(x(t)\right), \quad x(0) = x_0 \in \mathbb{R}^n$$

where F is uniformly convex: $HessF \geq \lambda I, \quad \lambda > 0$

- Unique equilibrium solution: x_∞ satisfies $\nabla F(x_\infty) = 0$.
  Stability: How fast do solutions $x(t)$ converge to x_∞?
- Suppose we can prove the (sharp) inequality:

$$F\left(x(t)\right) - F(x_\infty) \leq C_{sharp} \|\nabla F\left(x(t)\right)\|^2. \qquad (1)$$

Then

$$\frac{d}{dt}\left[F\left(x(t)\right) - F(x_\infty)\right] \leq -\frac{1}{C_{sharp}}\left[F\left(x(t)\right) - F(x_\infty)\right]$$

Illustration (continued)

- This implies:

$$F\left(x(t)\right) - F(x_\infty) \leq e^{-\frac{t}{C_{sharp}}}\left[F(x_0) - F(x_\infty)\right]$$

- If we can further prove the inequality:

$$\|x(t) - x_\infty\|^2 \leq K\left[F\left(x(t)\right) - F(x_\infty)\right] \qquad (2)$$

then we have the convergence (w.r.t. Euclidean norm) of $x(t)$ to x_∞ exponentially fast with the explicit rate

$$\|x(t) - x_\infty\|^2 \leq K e^{-\frac{t}{C_{sharp}}}\left[F(x_0) - F(x_\infty)\right]$$

It is easily to show that the inequalities (1) and (2) follow from the *convexity* of F, and $C_{sharp} = 1/2\lambda$. Then the sharp rate of convergence to equilibrium is λ.

Literature

- Questions 1 and 2 have been addressed in the literature under certain conditions on f and g (see [Gross, 75]; [Aubin, 76]; [Talenti, 76]; [Gidas, Ni & Nirenberg, 81]; [Li & Ni, 93]; [Serrin & Zou, 99]; [Serrin & Tang, 00]; [Tang, 01]; [Del-Pino & Dolbeault, 03]; ...).

- But the methods used so far to answer *Question 2* are quite complicated. The main idea has been to study the *much harder problem* of the uniqueness of solutions to the Euler-Lagrange equation of the variational problem (P), mainly because of the lack of convexity in the constraint $\int_{\mathbb{R}^n} g(u) = 1$; that is, the quasilinear pde

$$(PDE): \quad -\text{div}(|\nabla u|^{p-2}\nabla u) + f'(u) = \lambda g'(u),$$

where λ denotes the Lagrange multiplier for $\int_{\mathbb{R}^n} g(u) = 1$.

Traditional methods

- The traditional methods consists of two parts:
  - Part 1: **Reduction to 1-dimension.** Use the moving planes method, or the sliding method (see [Berestycki & Nirenberg, 91]), to show under suitable conditions on f and g, that the solutions to the PDE are *radially symmetric* about some point in $\mathbb{R}^n$ (see for e.g., [Gidas, Ni & Nirenberg, 79 & 81]; [Li & Ni, 93]; [Serrin & Zou, 99]; ...).
  - Part 2: **ODE analysis.** Prove the uniqueness of the radially symmetric solutions to the PDE by applying ODE techniques on the 1-dimensional analogue of the PDE (i.e., the corresponding ODE) (see [Serrin & Tang, 00]; [Tang, 01], ...). This analysis is very tedious, technical and not constructive.

New approach

• Here, we approach the problem *variationally*, by working directly on the variational problem $(\mathcal{P})$ rather than its associated PDE. Our proof is organized as follows:

- Step 1: **Symmetrization.** If f and g are even and nondecreasing (or in general, f is the difference of two monotone functions), and if $\lim_\infty g = \infty$, then the ground states of the system can be chosen *nonnegative and radially symmetric*.

  Proof.

  - *Nonnegativity*: $f,\ g$ are even $\Longrightarrow E(u) = E(|u|),\ \forall u \in S$.
  - *Radial symmetry* follows by rearrangement arguments: $\lim_\infty g = \infty$, and $f,\ g$ are nondecreasing $\Longrightarrow E(u^*) \leq E(|u|) = E(u),\ \forall u \in S$, where u^* is the symmetric decreasing rearrangement of u.

New approach (continued)

• From Step 1, we may study a reduced problem $(P)_r$ over the smaller space, S_r^+, of nonnegative radially symmetric (about the origin) functions satisfying $\int_{\mathbb{R}^n} g(u) = 1$.

- Step 2: **Optimal Transport.** We use a 1-dimensional optimal transport argument to study the uniqueness of the minimizers to $(P)_r$ on S_r^+. The key idea is to link (via a duality) our energy $E(u), u \in S_r^+$, to the Lyapunov function $H^F_{|x|^q/q}(\rho) := \int_{\mathbb{R}^n}(F(\rho) + \frac{|x|^q}{q}\rho)\mathrm{d}x$ of the convection-diffusion (or Fokker-Planck type) equation

$$\partial\rho/\partial t = \text{div}\,\left(\rho\nabla\left(F'(\rho) + |x|^q/q\right)\right)$$

  by transporting the probability density $g(u)$ to ρ; here $1/p + 1/q = 1$, $f = -[F + nP_F] \circ g$, $P_F(x) = xF'(x) - F(x)$.

Simple explanation of our method

To overcome the *lack of convexity* in the constraint $\int_{\mathbb{R}^n} g(u) = 1$ which prevents us from performing the classical uniqueness proof for the variational problem (P) (even though the minimizing functional $E(u)$ may be strictly convex), we introduce the change of variable $\rho = g(u)$. This transforms our non-convex admissible set S into the space of probability densities $\mathcal{P}(\mathbb{R}^n)$ which clearly is convex. With this change of variable, the "strictly convex" functional $E(u)$ becomes $J(\rho)$ which is NO MORE strictly convex. But fortunately, we regain in some sense, the strict convexity of $J(\rho)$ in the framework of *Optimal transport* via a duality between $J(\rho)$ and $H^F_{|x|^q/q}(\rho)$, since the functional $\mathcal{P}(\mathbb{R}^n) \ni \rho \mapsto H^F_{|x|^q/q}(\rho)$ is strictly *displacement convex*, and the new admissible set $\mathcal{P}(\mathbb{R}^n)$ is also displacement convex.

Duality by Optimal transport

Proposition 1 **[Cordero, Nazaret, Villani, 04]** $\&$ **[A, Ghoussoub, Kang, 04].** *Assume* $F : [0,\infty) \to \mathbb{R}$ *satisfies* $F(0) = 0$ *and* $x \mapsto x^n F(x^{-n})$ *is convex and non-increasing. Then for* $p, q > 1$ *s.t.* $1/q + 1/p = 1$*, and* $P_F(x) = xF'(x) - F(x)$*,*

$$\sup\left\{-H^F_{|x|^q/q}(\rho_1);\ \rho_1 \in \mathcal{P}(\mathbb{R}^n)\right\}$$

$$= \inf\left\{-H^{F+nP_F}(\rho_0) + \frac{1}{p}\int_{\mathbb{R}^n} \rho_0 |\nabla\left(F'(\rho_0)\right)|^p;\ \rho_0 \in \mathcal{P}(\mathbb{R}^n)\right\}$$

Furthermore, the $\rho_\infty \in \mathcal{P}(\mathbb{R}^n)$ *defined on its support by*

$$\nabla\left(F'(\rho_\infty) + |x|^q/q\right) = 0$$

is optimal in both variational problems.

Proof of Proposition 1

Let $\rho_0, \rho_1 \in \mathcal{P}(\mathbb{R}^n)$, and $T : \mathbb{R}^n \to \mathbb{R}^n$ be the optimal map transporting ρ_0 to ρ_1 with the quadratic cost, i.e. $T_\# \rho_0 = \rho_1$.

- *Geodesics*. The geodesic connecting ρ_0 and ρ_1 in $\mathcal{P}(\mathbb{R}^n)$ equipped with the Wasserstein distance is

$$[0,1] \ni t \mapsto \rho_t := (T_t)_\# \rho_0 \ \text{ where } \ T_t(x) := (1-t)x + tT(x)$$

- *Displacement convexity* [McCann, 97]: If $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ is convex and non-increasing, then

$$[0,1] \ni t \mapsto H^F(\rho_t) := \int_{\mathbb{R}^n} F\left(\rho_t(x)\right) \, \mathrm{d}x \ \text{ is convex.}$$

- *Displacement convexity inequality* [A, 02]:

$$H^F(\rho_1) - H^F(\rho_0) \geq \int_{\mathbb{R}^n} \rho_0 \nabla \left(F'(\rho_0)\right) \cdot (T(x) - x) \, \mathrm{d}x$$

Proof of Proposition 1 (continued)

- Can rewrite the convexity inequality as:

$$-H^F(\rho_1) \leq -H^{F+nP_F}(\rho_0) + \int_{\mathbb{R}^n} \rho_0 \nabla \left(F'(\rho_0)\right) \cdot T(x)\, \mathsf{d}x$$

- *Young inequality (with* $c(x) = |x|^q/q$ *i.e.,* $c^*(x) = |x|^p/p$*)*:

$$-\nabla \left(F'(\rho_0)\right) \cdot T(x) \leq \frac{1}{q}|T(x)|^q + \frac{1}{p}|\nabla \left(F'(\rho_0)\right)|^p$$

- Insert Young inequality into convexity inequality, and notice that $\frac{1}{q}\int_{\mathbb{R}^n} |T(x)|^q \rho_0 = \frac{1}{q}\int_{\mathbb{R}^n} |x|^q \rho_1 := H_{|x|^q/q}(\rho_1)$.
- If $\rho_0 = \rho_1$, then $T(x) = x$ and equality holds in Young inequality iff

$$-\nabla \left(F'(\rho_0)\right) = \nabla c\left(T(x)\right) = \nabla c(x) \Longleftrightarrow \nabla \left(F'(\rho_0) + |x|^q/q\right) = 0 \Box$$

Link duality and original problem (P)

Set $\rho_0 = g(u)$. If f, F, g are such that $f := -(F + nP_F) \circ g$ and $g|(F' \circ g)'|^p = 1$, then the $\inf$ problem in the *duality by optimal transport* coincides with our original problem (P),

$$\sup \left\{ -H^F_{|x|^q/q}(\rho);\ \rho \in \mathcal{P}(\mathbb{R}^n) \right\}$$

$$= \inf \left\{ E(u) := \frac{1}{p} \int_{\mathbb{R}^n} |\nabla u|^p + \int_{\mathbb{R}^n} f(u);\ \int_{\mathbb{R}^n} g(u) = 1 \right\}.$$

The $\inf$ problem defines the ground states for our pde

$$\text{div}(|\nabla u|^{p-2}\nabla u) - f'(u) + \lambda g'(u) = 0,$$

while the $\sup$ problem gives the equilibrium solution for F-P. equation,

$$\text{div}\ \left(\rho \nabla \left(F'(\rho) + |x|^q/q\right)\right) = 0$$

Link quasilinear pdes and (equi.) F-P.

This duality points to a correspondence between quasilinear pdes and (equilibrium) Fokker-Planck type eqs.

- If $p = q = 2$, $F(x) = -nx^{1-\frac{1}{n}}$, and $g(u) = u^{2^*}$ where $2^* := 2n/(n-2)$, then $f := -(F + nP_F) \circ g = 0$, and

$$\Delta u + u^{2^*-1} = 0 \longleftrightarrow \Delta(\rho^{1-\frac{1}{n}}) + \text{div}(\rho x) = 0.$$

- If $p = 2$, $F(x) = x \ln x$ and $g(u) = u^2$, we get:

$$\Delta u + u \ln u - u = 0 \longleftrightarrow \Delta \rho + \text{div}(\rho x) = 0$$

- If $p = 2$, $F(x) = x^\gamma/(\gamma - 1)$ where $\gamma := 1/s + 1/2$ and $2 < s < 2^*$, and $g(u) = u^s$, we get the correspondence

$$\Delta u - u^{s/2} + u^{s-1} = 0 \longleftrightarrow \Delta(\rho^\gamma) + \text{div}(\rho x) = 0.$$

Results

Uniqueness in the $\sup$ problem of the duality

- **Proposition 2** *If* $F(0) = 0$ *and* $x \mapsto x^n F(x^{-n})$ *is convex and non-increasing, then the* $\sup$ *problem of the duality (which is equivalently written as the* $\inf$ *problem),*

$$\inf \left\{ H^F_{|x|^q/q}(\rho) = \int_{\mathbb{R}^n} \left(F(\rho) + \frac{|x|^q}{q}\rho \right) \mathrm{d}x; \; \rho \in \mathcal{P}(\mathbb{R}^n) \right\}$$

has the unique solution $\rho_\infty \in \mathcal{P}(\mathbb{R}^n)$*, defined on its support by*

$$\nabla \left(F'(\rho_\infty) + |x|^q/q \right) = 0.$$

Moreover ρ_∞ *is radially symmetric about the origin.*

Results (continued)

Uniqueness in the $\inf$ problem of the duality

Denote by $\mathcal{P}_r(\mathbb{R}^n)$ the set of radially symmetric (about the origin) probability densities on $\mathbb{R}^n$.

- **Theorem 3**
  - *If $F(0)=0$ and $x \mapsto x^n F(x^{-n})$ is strictly convex, then ρ_∞ (the solution to the $\sup$ problem) is the unique minimizer of the $\inf$ problem over the set of radially symmetric (about the origin) $\rho \in \mathcal{P}_r(\mathbb{R}^n)$, i.e.,*

$$\inf\left\{-H^{F+nP_F}(\rho)+\frac{1}{p}\int_{\mathbb{R}^n}\rho|\nabla\left(F'(\rho)\right)|^p;\ \rho\in\mathcal{P}_r(\mathbb{R}^n)\right\}$$

  - *If $F(0)=0$ and $x \mapsto x^n F(x^{-n})$ is convex (but not strictly), say $F(x)=-nx^{1-\frac{1}{n}}$, i.e., $x^n F(x^{-n})=-nx$, then ALL the minimizers of the $\inf$ problem over $\mathcal{P}_r(\mathbb{R}^n)$ are of the form $\rho(x)=k^n\rho_\infty(kx)$, $k>0$.*

Results (continued)

Uniqueness in the original variational problem (P)

Denote by $(P)_r$ the reduced problem of (P) over the nonnegative radially symmetric (about the origin) functions,

$$(P)_r : \quad \inf_{u \in S_r^+} \left\{ E(u) := \frac{1}{p} \int_{\mathbb{R}^n} |\nabla u|^p + \int_{\mathbb{R}^n} f(u), \ \int_{\mathbb{R}^n} g(u) = 1 \right\}$$

Set $f = -(F + nP_F) \circ g$, $g|(F' \circ g)'|^p = 1$, $\rho = g(u)$ in Thm 3.

Corollary 4

- *If $x \mapsto x^n F(x^{-n})$ is strictly convex then the reduced problem $(P)_r$ has a unique solution $u_\infty \in S_r^+$, with $g(u_\infty) = \rho_\infty$. Hence all the minimizers $u \in S$ of the original problem (P) are $u(x) = \pm u_\infty(x - x_0)$, $x_0 \in \mathbb{R}^n$.*
- *If $x \mapsto x^n F(x^{-n})$ is convex (but not strictly), then all the minimizers of $(P)_r$ are s.t. $g(u(x)) = k^n \rho_\infty(kx)$. Hence all the minimizers of (P) are $g(|u|) = k^n \rho_\infty\,(k(x - x_0))$.*

Proof of Prop 2 (uniqueness in $\sup$)

- The uniqueness of the solution in the $\sup$ problem follows the classical method, with convexity replaced by displacement convexity. It is due to [McCann, 97]. Indeed,

  $$[0,1] \ni t \mapsto H^F_{|x|^q/q}(\rho_t) \text{ is strictly displacement convex}$$

  where ρ_t is the geodesic connecting two probability densities ρ_0, ρ_1 in $\mathcal{P}(\mathbb{R}^n)$ equipped with the Wasserstein distance; so $H^F_{|x|^q/q}(\rho_{\frac{1}{2}}) < \frac{1}{2}H^F_{|x|^q/q}(\rho_0) + \frac{1}{2}H^F_{|x|^q/q}(\rho_1)$.
- The existence of the solution ρ_∞ in the $\sup$ problem follows from the displacement convexity inequality, i.e.,

$$H^F_{|x|^q/q}(\rho_0) - H^F_{|x|^q/q}(\rho_1) \leq \int \rho_0 \nabla \left(F'(\rho_0) + |x|^q/q\right) \cdot (T(x) - x) \Box$$

Proof of Thm 3 (uniqueness in $\inf$)

The idea of the proof consists of establishing the inequality

$$-H^{F}_{|x|^q/q}(\rho_\infty) \le -H^{F+nP_F}(\rho) + \frac{1}{p}\int_{\mathbb{R}^n} \rho \left|\nabla \left(F'(\rho)\right)\right|^p, \ \forall \rho \in \mathcal{P}_r(\mathbb{R}^n)$$

while tracking on the way all the equations giving equalities, then later solve for ρ in the system of these equations. Only two ingredients will be used in the proof:

- *the strict convexity inequality for* $A(x) := x^n F(x^{-n})$ in 1d:

$$A(y) - A(x) \ge A'(x)(y-x) \text{ with equality iff } x = y.$$

- *the 1d-Young inequality with* $c(x) = |x|^a/a$, $a > 1$, *i.e.*, $c^*(x) = |x|^b/b$ *where* $1/a + 1/b = 1$:

$$xy \le c(x) + c^*(y) \text{ with equality iff } y = c'(x).$$

Outline of the proof of Thm 3

- Since $\rho_\infty(x)$ and $\rho(x)$ radially symmetric proba densities, then

$$\int_0^\infty \rho(r) r^{n-1} \mathrm{d}r = 1/(n\omega_n) = \int_0^\infty \rho_\infty(r) r^{n-1} \mathrm{d}r$$

Let $T(r)$ be the 1-dimensional optimal map transporting $\rho(r)r^{n-1}$ to $\rho_\infty(r)r^{n-1}$. Then $T(r)$ is C^1, nondecreasing, and

$$r^{n-1}\rho(r) = \left[T(r)^{n-1}\rho_\infty(T(r))\right] T'(r) \ \ \text{(1d Monge-Ampère eq.)}$$

- Starting from $-H^F_{|x|^q/q}(\rho_\infty)$ that we express in radial form in terms of $\rho_\infty(r)$, we establish a bunch of inequalities ($\leq$) to arrive to $-H^{F+nP_F}(\rho) + \frac{1}{p}\int_{\mathbb{R}^n} \rho |\nabla\left(F'(\rho)\right)|^p$, while collecting on our way all the cases of equalities.

Sketch of the proof

- Strict convexity inequality for $A(x) := x^n F(x^{-n})$ implies:

$$-H^F_{|x|^q/q}(\rho_\infty) \leq \mathcal{G}_1(\rho(r)) \text{ with equality iff } (\frac{T(r)}{r})^{n-1} T'(r) = 1$$

- Young inequality for $c(r) = r^n/n$ implies:

$$\mathcal{G}_1(\rho(r)) \leq \mathcal{G}_2(\rho(r)) \text{ with equality iff } T(r)/r = T'(r)$$

- Young inequality again with $c(r) = |r|^q/q$ implies:

$$\mathcal{G}_2(\rho(r)) \leq \mathcal{G}_3(\rho(r)) = -H^{F+nP_F}(\rho) + \frac{1}{p}\int_{\mathbb{R}^n} \rho |\nabla \left(F'(\rho)\right)|^p$$

with equality iff $-\frac{d}{dr}\left[F'(\rho(r))\right] = c'(T(r))$. $\square$

Sketch of the proof (continued)

- We conclude that the passage from $-H^{F}_{|x|^q/q}(\rho_\infty)$ to $-H^{F+nP_F}(\rho) + \frac{1}{p}\int_{\mathbb{R}^n} \rho|\nabla\left(F'(\rho)\right)|^p$ can be done in an optimal way iff equalities occur in all of the previous inequalities; this means that the following system of equations hold,

$$\left\{\begin{array}{lcl} (\frac{T(r)}{r})^{n-1}T'(r) & = & 1 \\ \frac{T(r)}{r} & = & T'(r) \\ -\frac{d}{dr}\left[F'(\rho(r))\right] & = & c'(T(r)) \end{array}\right.$$

  The first 2 equations give that $T(r) = r$, and the last equation confirm that $\rho(r) = \rho_\infty(r)$.

- If $A(x)$ is convex but not strictly (e.g., $F(x) = -nx^{1-1/n}$), the first equation is not needed. Then $T(r) = kr$, $k > 0$.

Application to geometric inequalities

- If $F(x) = -nx^{1-1/n}$, we get the sharp Sobolev inequalities with the explicit sharp constant, and ALL the optimal functions are of the form $Cu_\infty(k(x-x_0))$.
- If $F(x) = x^\gamma/(\gamma-1)$ for some conveniently chosen $1 \neq \gamma > 1-1/n$, we get the sharp constant and ALL the optimal functions of some Gagliardo-Nirenberg inequalities, which are $Cu_\infty(k(x-x_0))$.
- If $F(x) = x \ln x$, we get the sharp logarithmic Sobolev inequality of [Gross, 75], and the optimal functions are the constant functions.

Conclusion: *Optimal Transport not only allows to compute the sharp constants in many geometric inequalities, but it also enables to identify all the optimal functions in these inequalities in a more structural way*.