# Spline Element Method for Monge-Ampère equations

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Recent developments in nonlinear analysis and applications, Cotonou 2010

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# Outline

Fully nonlinear $F(x,u,Du,D^2u)=0, x\in\Omega$.
We consider $F(x,D^2u(x))=0$.
$F$ is uniformly elliptic if, $\exists \lambda,\Lambda>0$ / for all $M$ in $\mathbb{S}$, $x\in\Omega$

$$\lambda||N||\leq F(x,M+N)-F(x,M)\leq \Lambda||N||, \forall N\geq 0,$$
$$\lambda|\xi|^2\leq \frac{\partial F}{\partial m_{ij}}(x,M)\xi_i\xi_j\leq \Lambda|\xi|^2, \forall \xi\in\mathbb{R}^n.$$

Main examples : Monge-Ampère and Pucci equations
Prescribed Gauss curvature $\det D^2u-K(x)(1+|Du|^2)^{\frac{n+2}{2}}=0$
Bellman equation $F(x,D^2u):=\inf_{\alpha\in\mathcal{A}}(L_\alpha u(x)-f_\alpha(x))=0$
Isaacs equation $F(x,D^2u):=\sup_\beta \inf_\alpha(L_{\alpha,\beta}u(x)-f_{\alpha,\beta}(x))=0$

## Monge-Ampère Equation

$$\det D^2 u = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega,\ f > 0$$

Appears in mass transportation problems, kinetic theory, meteorology, fluid mechanics, nonlinear elasticity, antenna design, material sciences and mathematical finance.

## Pucci Equations

$$M^+[u] := \alpha \Delta u + (1 - n\alpha)\lambda^+(D^2 u) = f$$
$$M^-[u] := \alpha \Delta u + (1 - n\alpha)\lambda^-(D^2 u) = f, 0 < \alpha \leq \frac{1}{n}.$$

Appear in stochastic control where the control variable is the diffusion coefficient.
Pucci extremal operators play an important role in the theory of fully nonlinear equations.

# Monge-Ampère

Concept of generalized solutions and viscosity solutions

Alexandrov-Bakelman solution equivalent to viscosity solution for $f$ continuous

Theoretical convergence of numerical solutions not easy.

Geometric, Oliker and Prussner 1988 for Alexandrov solution

Oberman 2008 for viscosity solution

Some very simple difference methods, Benamou, Froese and Oberman 2010
Partial convergence theory for finite elements, Bohmer 2008

"One of the outstanding challenges to mathematical theory and ingenuity is the construction of algorithms which provide numerical answers to numerical questions . . . The effective use of the digital computer requires a combination of superior mathematical training and mathematical ingenuity"
Richard Bellman, preface to the book
The method of quasi-reversibility, Applications to partial differential equations by Lattès and Lions

# Finite element based approaches

Dean and Glowinski, 2003, 2004, 2006. Augmented Lagrangian and Least Squares.

Feng and Neilan, 2007, 2008, 2009. Vanishing moment methodology.

$$-\epsilon\Delta^2 u + \det D^2 u = f, \text{ in } \Omega, \quad u = g, \; \Delta u = \epsilon^2 \text{ on } \partial\Omega.$$

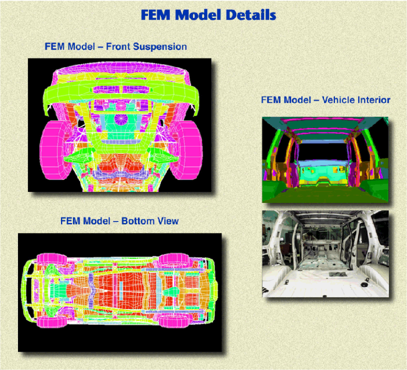
Benamou, Froese and Oberman, 2009. Fixed point iteration for solving $u = T(u)$ where

$$T(u) = \Delta^{-1}(\sqrt{(\Delta u)^2 + 2(f - \det D^2 u)})$$

Bohmer 2008, projection.

## Finite element implementation with Lagrange multipliers

The finite element method is the most widely used method for solving numerically partial differential equations
A collection of methods falls under the designation f.e.m.




http ://www.epm.ornl.gov/SC98/car.html

## Model problem

$$\begin{cases} -\Delta u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega \end{cases}$$

where $\partial\Omega$ will denote the boundary of the bounded domain $\Omega$ and $\Delta$ denotes the Laplace operator, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.
Green's identity

$$\int_{\partial\Omega} (-\operatorname{div} \nabla u) v \ dx = \int_{\partial\Omega} \nabla u \cdot \nabla v \ dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v.$$

Find $u$ in $H_0^1(\Omega)$ such that

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} fv, \quad \forall v \in H_0^1(\Omega),$$

## Biharmonic equation

$$\begin{cases} \Delta^2 u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega \\ \frac{\partial u}{\partial \mathbf{n}} &= h \text{ in } \partial\Omega, \end{cases}$$

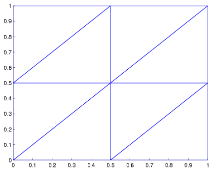
$$\int_\Omega \Delta u \, \Delta v \, dx = \int_\Omega f \, v, \ \forall v \in H_0^2(\Omega), \qquad (1)$$

**Abstract variational problem** Find $u \in V$ such that

$$a(u, v) = \langle l, v \rangle \quad \text{for all } v \in V$$

Lax Milgram lemma
If $a$ is symmetric, $u$ is the unique minimizer in $V$ of the functional $J(v) = 1/2a(v, v) - \langle l, v \rangle$.

# Discrete approximations




Find $u_h \in V_h$ such that

$$a(u_h, v) = \langle l, v \rangle \quad \text{for all } v \in V_h$$

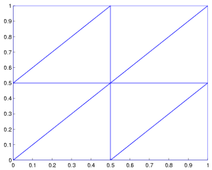
Equivalently, $u_h$ is the unique minimizer in $V_h$ of $J_h(u) = 1/2a(u,u) - \langle l, u \rangle$

Cea's lemma

$$||u - u_h||_V \leq C \min_{v \in V_h} ||u - v||_V$$

for a constant $C$ independent of h.

## Discrete approximations

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## Requirements for conforming approximations

Let $k \geq 1$ and suppose $\Omega$ is bounded. Then a piecewise infinitely differentiable function $v : \overline{\Omega} \to \mathbf{R}$ belongs to $H^k(\Omega)$ if and only if $v \in C^{k-1}(\overline{\Omega})$.

Even in two dimensions, conforming finite element spaces can be very complicated and there are no satisfactory answer in three dimensions.

Nonconforming approximations are not very popular.

Stokes equations Find $(\mathbf{u}, p) \in H^1(\Omega)^n \times L_0^2(\Omega)$ such that

$$\begin{cases} -\nu\, \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega \end{cases}$$

Equilibrium conditions are very difficult to enforce in the finite element method

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Equilibrium conditions are very difficult to enforce in the finite element method

Conforming finite element implementation with Lagrange multipliers
Start with a representation of a piecewise discontinuous polynomial as a vector in $\mathbb{R}^N$
Express boundary conditions and constraints including global continuity or smoothness conditions as linear relations
Discretize the equation. Then enforce boundary conditions and constraints using Lagrange multipliers
Contrast with other approaches where Lagrange multipliers are introduced before discretization
Leads to saddle point problems which are solved by an augmented Lagrangian algorithm (sequences of linear equations with size $N \times N$)
Main advantages Flexible. No need to implement basis functions. Constraints and smoothness are enforced. Variable degree approximation.
Possible disadvantage Large matrix sizes

"spline" because we use Bernstein basis representations very convenient to express smoothness conditions

# Splines on Plane Triangulations

Let $d \geq 1$ and $r \geq 0$

$$S_d^r(\Omega) = \{p \in C^r(\Omega),\ p_{|t} \in P_d,\ \forall t \in \mathcal{T}\}.$$

$$B_{ijk}^d(v) = \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k, \qquad i+j+k=d.$$

The set $\mathcal{B}^d = \{B_{ijk}^d(x,y), i+j+k=d\}$ is a basis for $P_d$.

$$s_{|T} = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}^d,$$

## Interpolation

On the triangle $T = \langle v_1, v_2, v_3 \rangle$ at $\xi_{ijk} = \frac{iv_1 + jv_2 + kv_3}{d}$

On the edge $\langle v_1, v_2 \rangle$ at $\xi_{ij} = \frac{iv_1 + jv_2}{d}$

$$\sum_{i+j=d} \widetilde{c}_{ij} \widetilde{B}_{ij}^d(v), \qquad \widetilde{B}_{ij}^d = \frac{d!}{i!j!} b_1^i b_2^j.$$

$$p = \sum_{i+j+k=d} c_{ijk} B_{ijk}^d, \qquad q = \sum_{i+j=d} c_{ij0} B_{ij0}^d$$

$$Rc = c_b$$

Derivatives

$D_i c$, $i = 1, 2$ encode respectively the $B$-net of $\frac{\partial s}{\partial x_i}$.

Integration

$$\int_\Omega pq = c^T I d$$

Smoothness conditions

There's a $(l, N)$ matrix $H$ such that $s$ is in $C^r(\Omega)$ if and only if

$$Hc = 0.$$

# Features of the Spline Method

Find $u \in W$ such that

$$a(u, v) = \langle l, v \rangle \quad \text{for all } v \in V$$

- Find $u$ in $H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega),$$

- Find $u \in H^2(\Omega)$, $u = g$ on $\partial\Omega$ such that

$$\int_\Omega (\text{cof } D^2 u) Du \cdot Dv \ dx = -n \int_\Omega fv \ dx, \quad \forall v \in H_0^1(\Omega).$$

$$V_h = \{c \in \mathbf{R}^N, Rc = 0\},$$

$$W_h = \{c \in \mathbf{R}^N, Rc = G\}$$

The condition $a(u, v) = \langle l, v \rangle$ for all $v \in V_h$ becomes

$$K(c, c)d = L^T d \quad \forall d \in V_h, \text{ that is for all } d \text{ with } Rd = 0$$

$$K(c, c) + \lambda^T R = L^T.$$

$$\begin{bmatrix} K^T & R^T \\ R & 0 \end{bmatrix} \begin{bmatrix} c \\ \lambda \end{bmatrix} = \begin{bmatrix} L \\ G \end{bmatrix}$$

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Recall Cea's lemma $||u - u_h||_V \leq C \min_{v \in V_h} ||u - v||_V$

Assume $V_h$ is a spline space $S^r_d(\mathcal{T}_h)$

Bivariate splines For $d \geq 3r + 2$ and $0 \leq m \leq d$ and for $0 \leq k \leq m$.

$$|f - Qf|_{p,k} \leq Ch^{m+1-k} |f|_{p,m+1}$$

The constant $C$ depends on the smallest angle in $\triangle$

Trivariate splines For $0 \leq k \leq d$, $f \in W^{d+1}_p(\Omega)$

$$|f - Qf|_{p,k} \leq Ch^{d+1-k} |f|_{p,d+1}$$

The constant $C$ depends on the shape parameter $\sigma_K = h_K / \rho_K$

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# Advantages of the method

- Can be applied to a wide range of PDEs in science and engineering in both two and three dimensions.
- Constraints and Smoothness are enforced exactly and there is no need to implement basis functions with the required properties. Particularly suitable for higher order PDEs.
- No inf-sup condition
- One gets in a single implementation approximations of variable order.
- The mass and stiffness matrices are assembled easily and this can be done in parallel.

# Possible Disadvantages

- Large size matrices for 3D problems and high order approximations
- $$\begin{bmatrix} K^T & R^T \\ R & 0 \end{bmatrix} \begin{bmatrix} c \\ \lambda \end{bmatrix} = \begin{bmatrix} L \\ G \end{bmatrix} \quad \begin{pmatrix} K^T & R^T \\ R & -\mu I \end{pmatrix} \begin{bmatrix} \mathbf{c}^{(l+1)} \\ \lambda^{(l+1)} \end{bmatrix} = \begin{bmatrix} L \\ G - \mu\lambda^{(l)} \end{bmatrix}$$

  Computing $c^{(1)}$ from $\lambda^{(0)}$, one solves

  $$(K^T + \frac{1}{\mu} R^T R) c^{(l+1)} = K^T c^{(l)} + \frac{1}{\mu} R^T G, \quad l = 1, 2, \dots$$

  $$\|c - c^{(l+1)}\| \leq C\mu \|c - c^{(l)}\|$$

# Examples of Applications


- G. Awanou, M. J. Lai and P. Wenston, The multivariate spline method for numerical solutions of partial differential equations and scattered data fitting, Wavelets and Splines : Athens 2005, pp. 24–74.
- V. Baradmize and M.J. Lai, Spherical spline solution to a PDE on sphere, Wavelets and Splines : Athens 2005, pp. 75–92.
- G. Awanou and M. J. Lai, Trivariate spline approximations of 3D Navier-Stokes equations, Math. Comp. **74** (2005), pp. 585–601.
- G. Awanou, Robustness of a spline element method with constraints, Journal of Scientific Computing, **36**, 3, (2009) pp. 421–432.
- Xian-Liang Hu, Dan-Fu Han, Ming-Jun Lai, Bivariate Splines of Various Degrees for Numerical Solution of Partial Differential Equations, SIAM Journal of Scientific Computing, 29(2007) 1338–1354

Test 1 : a smooth solution
Test 2 : solution not in $H^2(\Omega)$
Test 3 : no exact solution. Expects a convex solution.

Method 1 Newton's method.
Dean and Glowinski in Numerical Methods for fully nonlinear equations of the Monge-Ampère type "Newton's and conjugate gradients methods may be well-suited for the solution of . . . combines the difficulty of both harmonic and bi-harmonic problems, making the approximation a delicate matter, albeit solvable . . . If . . . has no solution, we can expect the divergence of the Newton . . . "
Method 2 Vanishing moment is analogue of a singular perturbation problem

$$\epsilon\Delta^2 u - \Delta u = f \text{ in } \Omega, u = 0, \frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega$$

Method 3 Iterative method

Put $F(u) = \det D^2 u - f$. Then $F(u) = 1/n \operatorname{div}\left((\operatorname{cof} D^2 u) Du\right) - f$.
$F'(u)w = \operatorname{div}\left((\operatorname{cof} D^2 u) Dw\right)$.
Variational formulation Find $u \in H^2(\Omega), u = g$ on $\partial\Omega$ such that

$$-\frac{1}{n}\int_\Omega (\operatorname{cof} D^2 u) Du \cdot Dv \; dx = \int_\Omega fv \; dx, \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega).$$

Algorithm 1 Newton's method

$$F'(u_k)(u_{k+1} - u_k) = -F(u_k)$$

$$\operatorname{div}\left((\operatorname{cof} D^2 u_k)(Du_{k+1} - Du_k)\right) = -\frac{1}{n} \operatorname{div}\left((\operatorname{cof} D^2 u_k) Du_k\right) + f,$$

$$\int_\Omega (\operatorname{cof} D^2 u_k) Du_{k+1} \cdot Dv \; dx = \frac{n-1}{n} \int_\Omega (\operatorname{cof} D^2 u_k) Du_k \cdot Dv \; dx - \int_\Omega fv \; dx, \quad \forall v \in V_0.$$

Algorithm 2 (Newton's method of vanishing moment formulation Feng-Neilan)

$$\epsilon \int_\Omega \Delta u_{k+1} \Delta v \, dx + \int_\Omega (\mathrm{cof}\, D^2 u_k) D u_{k+1} \cdot Dv \;\, dx = \frac{n-1}{n} \int_\Omega (\mathrm{cof}\, D^2 u_k) D u_k \cdot Dv \;\, dx$$
$$+\epsilon^3 \int_{\partial\Omega} \frac{\partial v}{\partial n} \, ds - \int_\Omega f v \;\, dx, \quad \forall v \in H^2(\Omega) \cap H^1_0(\Omega)$$

with $u_{k+1} = g$ on $\partial\Omega$

Algorithm 3 (Iterative method of Benamou-Froese-Oberman)

$$\Delta u_{k+1} = \sqrt{(\Delta u_k)^2 + 2(f - \det D^2 u_k)}$$

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# Analysis of algorithms

Based on the work of Loeper and Rapetti Newton's method can be shown to converge based on global Schauder estimates due to Wang. For a $C^3$ domain, $g \in C^3(\overline{\Omega})$, inf $f > 0$, and $f \in C^\alpha$ for some $\alpha \in (0, 1)$

# 2D Newton's method

| d | $L^2$ norm | $H^1$ norm | $H^2$ norm |
|---|---|---|---|
| 3 | $1.0610\ 10^{-3}$ | $1.1101\ 10^{-2}$ | $1.6383\ 10^{-1}$ |
| 4 | $3.5127\ 10^{-5}$ | $4.8553\ 10^{-4}$ | $9.0596\ 10^{-3}$ |
| 5 | $4.1572\ 10^{-6}$ | $6.5142\ 10^{-5}$ | $1.9364\ 10^{-3}$ |
| 6 | $1.9685\ 10^{-7}$ | $3.6401\ 10^{-6}$ | $1.4774\ 10^{-4}$ |
| 7 | $2.2699\ 10^{-8}$ | $4.1498\ 10^{-7}$ | $2.2424\ 10^{-5}$ |
| 8 | $1.2430\ 10^{-9}$ | $2.2586\ 10^{-8}$ | $1.5479\ 10^{-6}$ |
| Rate | $10^{3.7312}\ d^{-13.5794}$ | $10^{3.7312}\ d^{-12.9970}$ | $10^{4.8498}\ d^{-11.3673}$ |

Test 1 : $u(x,y) = e^{(x^2+y^2)/2}$ $h = 1/2$

| d | $L^2$ norm | $H^1$ norm | $H^2$ norm |
|---|---|---|---|
| 3 | $1.2809\ 10^{-4}$ | $2.6554\ 10^{-3}$ | $8.9587\ 10^{-2}$ |
| 4 | $1.6278\ 10^{-6}$ | $4.5619\ 10^{-5}$ | $1.7395\ 10^{-3}$ |
| 5 | $1.1531\ 10^{-7}$ | $2.3916\ 10^{-6}$ | $1.3444\ 10^{-4}$ |
| 6 | $1.7705\ 10^{-9}$ | $6.8506\ 10^{-8}$ | $5.5403\ 10^{-6}$ |
| 7 | $1.4548\ 10^{-10}$ | $3.7545\ 10^{-9}$ | $3.9490\ 10^{-7}$ |
| 8 | $8.1014\ 10^{-12}$ | $5.3353\ 10^{-10}$ | $7.2159\ 10^{-8}$ |
| Rate | $10^{4.3451}\ d^{-16.8127}$ | $10^{5.2200}\ d^{-15.9649}$ | $10^{5.9692}\ d^{-14.4722}$ |

Test 1 : $u(x,y) = e^{(x^2+y^2)/2}$ $h = 1/4$

| $\epsilon$ | $L^2$ norm | $H^1$ norm | $H^2$ norm |
|---|---|---|---|
| $10^{-3}$ | $1.7271 10^{-3}$ | $2.0910 10^{-2}$ | $8.6338 10^{-1}$ |
| $10^{-4}$ | $1.8563 10^{-4}$ | $3.5581 10^{-3}$ | $2.0050 10^{-1}$ |
| $10^{-5}$ | $1.8917 10^{-5}$ | $4.0700 10^{-4}$ | $2.4119 10^{-2}$ |
| $10^{-6}$ | $1.8182 10^{-6}$ | $4.0775 10^{-5}$ | $2.4388 10^{-3}$ |
| $10^{-7}$ | $1.2441 10^{-7}$ | $4.0951 10^{-6}$ | $2.4949 10^{-4}$ |
| $10^{-8}$ | $1.0119 10^{-7}$ | $2.2962 10^{-6}$ | $1.3029 10^{-4}$ |
| $10^{-9}$ | $1.1384 10^{-7}$ | $2.3790 10^{-6}$ | $1.3382 10^{-4}$ |
| $10^{-10}$ | $1.1516 10^{-7}$ | $2.3903 10^{-6}$ | $1.3438 10^{-4}$ |
| $10^{-11}$ | $1.1530 10^{-7}$ | $2.3914 10^{-6}$ | $1.3443 10^{-4}$ |
| $10^{-14}$ | $1.1531 10^{-7}$ | $2.3916 10^{-6}$ | $1.3444 10^{-4}$ |
| Reduced | $1.1531\ 10^{-7}$ | $2.3916\ 10^{-6}$ | $1.3444\ 10^{-4}$ |

Comparison of Newton's method with (Newton) vanishing moment method Test 1, $h = 1/4$, $d = 5$. No boundary layers issue with spline element method.

| $h$ | $n_{\text{it}}$ | $L^2$ norm | $H^1$ norm | $H^2$ norm |
|---|---|---|---|---|
| $1/2^1$ | 41 | $2.8275\ 10^{-6}$ | $6.1372\ 10^{-5}$ | $1.8845\ 10^{-3}$ |
| $1/2^2$ | 37 | $5.4642\ 10^{-8}$ | $2.1971\ 10^{-6}$ | $1.2972\ 10^{-4}$ |
| $1/2^3$ | 38 | $8.3164\ 10^{-10}$ | $7.2252\ 10^{-8}$ | $8.4790\ 10^{-6}$ |
| $1/2^4$ | 37 | $2.7871\ 10^{-9}$ | $1.4089\ 10^{-8}$ | $1.0809\ 10^{-6}$ |

BFO iterative method for Test 1, $d = 5$

| $d$ | $n_{\text{it}}$ | $L^2$ norm | $H^1$ norm | $H^2$ norm |
|---|---|---|---|---|
| 3 | 38 | $2.1905\ 10^{-6}$ | $1.3368\ 10^{-4}$ | $1.9600\ 10^{-2}$ |
| 4 | 39 | $5.1317\ 10^{-9}$ | $5.9766\ 10^{-7}$ | $7.7660\ 10^{-5}$ |
| 5 | 35 | $2.2563\ 10^{-9}$ | $1.2616\ 10^{-8}$ | $1.0166\ 10^{-6}$ |
| 6 | 34 | $1.2775\ 10^{-9}$ | $1.3206\ 10^{-8}$ | $2.9168\ 10^{-6}$ |
| 7 | 31 | $2.8787\ 10^{-9}$ | $3.5309\ 10^{-8}$ | $8.2791\ 10^{-6}$ |
| 8 | 26 | $2.1192\ 10^{-8}$ | $2.6174\ 10^{-7}$ | $1.7105\ 10^{-5}$ |

BFO iterative method for Test 1, $h = 1/2^4$

Newton's method diverges for this test $d = 3$

| $h$ | $L^2$ norm | $H^1$ norm |
|---|---|---|
| $1/2^1$ | $2.195410^{-2}$ | $1.640910^{-1}$ |
| $1/2^2$ | $3.609710^{-3}$ | $6.140510^{-2}$ |
| $1/2^3$ | $1.068510^{-3}$ | $4.097810^{-2}$ |
| $1/2^4$ | $5.083810^{-3}$ | $2.804810^{-1}$ |
| $1/2^5$ | $2.579710^{+3}$ | $2.268810^{+5}$ |
| $1/2^6$ | $1.845210^{+4}$ | $3.592210^{+6}$ |

Test 2 : $u(x, y) = -\sqrt{2 - x^2 - y^2} \notin H^2(\Omega)$ $d = 3$

Damped Newton's method, $d = 3, \tau = 50$.
Results here better than Glowinski's

$$F'(u_k)(u_{k+1} - u_k) = -\frac{1}{\tau} F(u_k), \tau \geq 1$$

Convergence for smooth solutions on smooth domains for some $\tau \geq 1$

| $h$ | $L^2$ norm | $H^1$ norm |
|---|---|---|
| $1/2^1$ | $2.1173\ 10^{-2}$ | $1.5928\ 10^{-1}$ |
| $1/2^2$ | $2.6709\ 10^{-3}$ | $3.4032\ 10^{-2}$ |
| $1/2^3$ | $2.6400\ 10^{-4}$ | $7.7037\ 10^{-3}$ |
| $1/2^4$ | $2.5420\ 10^{-5}$ | $2.1618\ 10^{-3}$ |

Vanishing moment Feng-Neilan $\epsilon = 10^{-3}$, $d = 5$

| $h$ | $L^2$ norm | $H^1$ norm |
|---|---|---|
| $1/2^1$ | $7.6680 10^{-3}$ | $7.4491 10^{-2}$ |
| $1/2^2$ | $1.4536 10^{-3}$ | $3.9244 10^{-2}$ |
| $1/2^3$ | $9.8727 10^{-3}$ | $2.5112 10^{-1}$ |
| $1/2^4$ | $5.6819 10^{-3}$ | $2.4927 10^{-1}$ |
| $1/2^5$ | $1.9830\ 10^{+4}$ | $1.1812\ 10^{+6}$ |

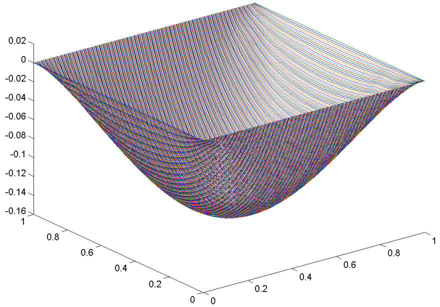
Vanishing moment Feng-Neilan $\epsilon = 10^{-2}$, $d = 5$

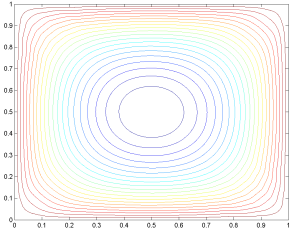
| $h$ | $L^2$ norm | $H^1$ norm |
|---|---|---|
| $1/2^1$ | $7.8254 10^{-3}$ | $9.3184 10^{-2}$ |
| $1/2^2$ | $1.0646 10^{-2}$ | $9.5201 10^{-2}$ |
| $1/2^3$ | $1.1306 10^{-2}$ | $9.6154 10^{-2}$ |
| $1/2^4$ | $1.1500 10^{-2}$ | $9.1336 10^{-2}$ |
| $1/2^5$ | $1.1625 10^{-2}$ | $8.7785 10^{-2}$ |
| $1/2^6$ | $1.1681 10^{-2}$ | $8.5632 10^{-2}$ |

BFO iterative method $d = 3$, $r = 0$ and $r = 1$

| $h$ | $n_{\text{it}}$ | $L^2$ norm | $H^1$ norm |
|---|---|---|---|
| $1/2^1$ | 50 | $2.3921\ 10^{-1}$ | 1.1900 |
| $1/2^2$ | 159 | $1.2585\ 10^{-1}$ | $7.1292\ 10^{-1}$ |
| $1/2^3$ | 151 | $1.0341\ 10^{-1}$ | $6.4299\ 10^{-1}$ |
| $1/2^4$ | 160 | $9.6031\ 10^{-2}$ | $6.2088\ 10^{-1}$ |
| $1/2^5$ | 199 | $9.4551\ 10^{-2}$ | $6.2453\ 10^{-1}$ |
| $1/2^6$ | 8 | $1.6977\ 10^{-2}$ | $2.2925\ 10^{-1}$ |

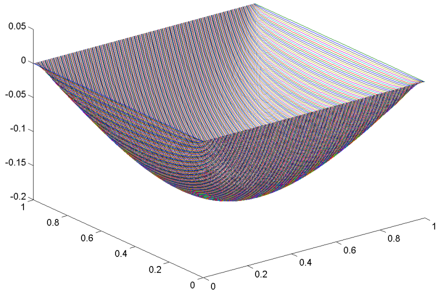
# No smooth solution Damped Newton's method

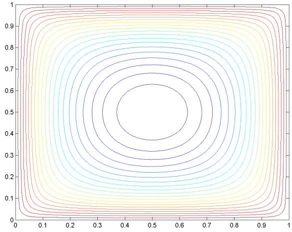



Test 3 : $f(x,y) = 1$ and $g(x,y) = 0$ on $h = 1/2^4, d = 3, r = 1, \tau = 100.$

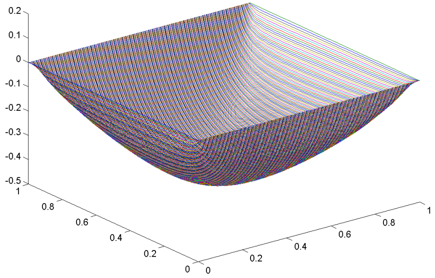
# No smooth solution Vanishing moment Feng-Neilan

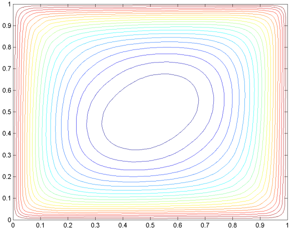



Test 3 : $f(x, y) = 1$ and $g(x, y) = 0$ on $h = 1/2^4, d = 3$.

# No smooth solution Benamou-Froese-Oberman




Test 3 : $f(x, y) = 1$ and $g(x, y) = 0$ on $h = 1/2^4, d = 3$.

# 3D Newton's method

| d | $L^2$ norm | $H^1$ norm | $H^2$ norm |
|---|---|---|---|
| 3 | $5.7401\ 10^{-3}$ | $3.9308\ 10^{-2}$ | $3.8805\ 10^{-1}$ |
| 4 | $4.9653\ 10^{-4}$ | $4.7672\ 10^{-3}$ | $7.6089\ 10^{-2}$ |
| 5 | $6.9025\ 10^{-5}$ | $7.7162\ 10^{-4}$ | $1.5425\ 10^{-2}$ |
| 6 | $1.0005\ 10^{-5}$ | $1.0310\ 10^{-4}$ | $2.5992\ 10^{-3}$ |
| 7 | $1.8956\ 10^{-6}$ | $1.7983\ 10^{-5}$ | $4.8870\ 10^{-4}$ |
| Rate | $10^{2.3278}\ d^{-9.4276}$ | $10^{3.0318}\ d^{-9.0347}$ | $10^{3.4794}\ d^{-7.8305}$ |

$$u(x,y,z) = e^{(x^2+y^2+z^2)/3} \text{ on } \mathcal{I}_1$$

| d | $L^2$ norm | $H^1$ norm | $H^2$ norm |
|---|---|---|---|
| 3 | $4.7273\ 10^{-3}$ | $2.660610^{-2}$ | $1.782910^{-1}$ |
| 4 | $6.500110^{-5}$ | $9.660910^{-4}$ | $2.218310^{-2}$ |
| 5 | $2.139910^{-6}$ | $4.560010^{-5}$ | $1.582510^{-3}$ |
| 6 | $1.381510^{-6}$ | $2.236810^{-5}$ | $6.092010^{-4}$ |
| 7 | $1.528310^{-7}$ | $1.675310^{-6}$ | $5.618910^{-5}$ |
| Rate | $10^{3.1038}\ d^{-11.8706}$ | $10^{3.6366}\ d^{-11.0367}$ | $10^{3.7955}\ d^{-9.3039}$ |

$$u(x,y,z) = e^{(x^2+y^2+z^2)/3} \text{ on } \mathcal{T}_2$$

3D version of BFO

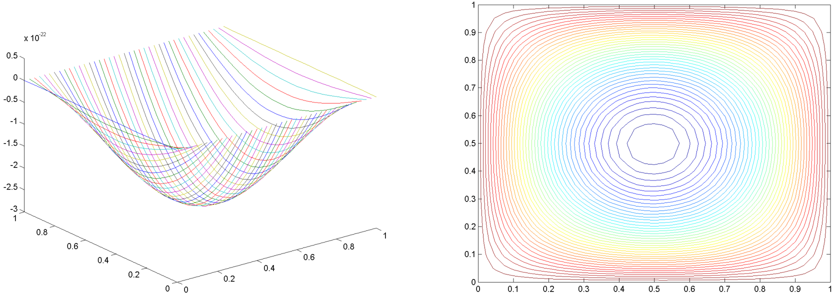
$$\Delta u_{k+1} = ((\Delta u_k)^3 + 9(f - \det D^2 u_k))^{\frac{1}{3}},$$

with $u_{k+1} = g$ on $\partial\Omega$.

$$\det D^2 u \leq \frac{1}{n^n}(\Delta u)^n,$$

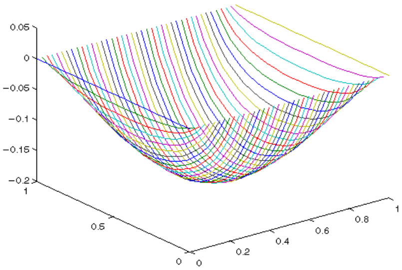
Enforces partial convexity since $\Delta u_k \geq 0$

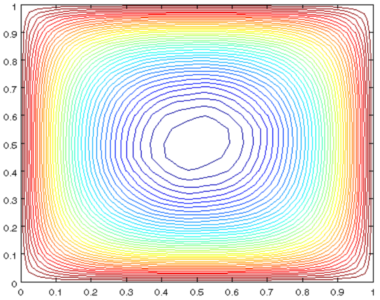
# No smooth solution for f=1, g=0



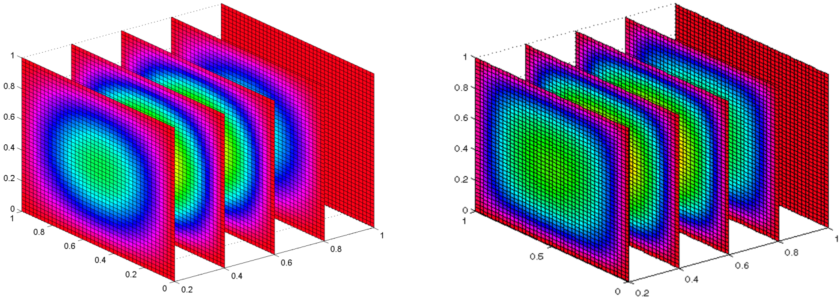

Test 2 : $f(x, y, z) = 1$ and $g(x, y, z) = 0$ on $\mathcal{I}_2, d = 3, r = 2, \epsilon = 10^{-3}$. Vanishing moment

# No smooth solution for f=1, g=0



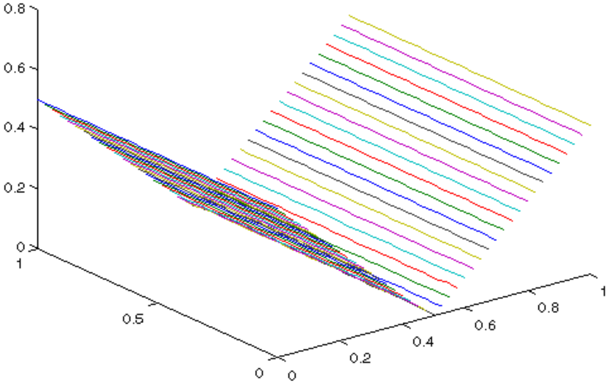




Test 2 : $f(x, y, z) = 1$ and $g(x, y, z) = 0$ on $\mathcal{I}_2$, $d = 5$, $r = 1$. 3D BFO




Slices in the $x-$direction on Domain 2 and $\mathcal{I}_3, d = 3$ , Vanishing moment $r = 2, \epsilon = 10^{-5}$ and BFO $d = 5, r = 1$

# f=0, $g(x, y, z) = |x - 1/2|$ 3D BFO



# Concluding remarks

For homogeneous boundary conditions, i.e. for $u = 0$ on $\partial\Omega$, i.e. for $u(x, y) = \pm x(1 - x)y(1 - y)exp(x^2 + y^2)$, both concave and convex solutions

Newton's method : initial guess $\pm u_0$
Vanishing moment (Feng-Neilan) : $\pm\epsilon$
Benamou-Froese-Oberman : $u_{k+1} = \pm T(u_k)$

Last two perform (to some extent) when there is no smooth solution
Vanishing moment (with Newton) gives Newton's method result for $\epsilon = 10^{-9}$

# Future directions

Proven convergence methods

Benamou-Brenier " It follows from this theoretical result that a natural computational solution of the $L^2$ MKP is the numerical resolution of the Monge-Ampère equation" . . . "Unfortunately, this fully nonlinear second-order elliptic equation has not received much attention from numerical analysts and, to the best of our knowledge, there is no efficient finite-difference or finite-element methods, comparable to those developed for linear second-order elliptic equations (such as fast Poisson solvers, multigrid methods, preconditioned conjugate gradient methods,. . . ). "

Connections with optimal transportation in particular work of Barrett and Prigozhin

Other engineering applications

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