Transport and Topological Optimization Problems

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I: Sand Transport by Homogenization tools

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Introduction

Roughly speaking, the models in use essentially couple an equation for the fluid fields (Navier-Stocks or shallow water equations) to an equation describing sand transport on the seabed. Those methods were used with success in De-Vriend , Engelund and Hansen , Kennedy , Blondeau, Dawson *et al.* , Johns Soulsby and Chesher , Idier and Idier, Astruc and Hulsher .

A careful watch reveals that the use of numerical simulation for the understanding of dune dynamics within tide-influenced environment is essentially not efficient.

The reason why is that tide oscillation generally prompts a coming and going of large sand volumes having a very weak resulting effect on dune evolution.

As a consequence, questions concerning dune morphodynamics or stability have to be considered over large periods of time, making the computation cost expensive.

Since many dune fields are present in strong tide region (English Channel, Celtic Sea, Irish Sea, North Sea, etc.) the setting out of methods to tackle dune morphodynamics in tide influenced environments is an important challenge.

The aim of this work is to carry out modeling methods and asymptotical methods for this.

More precisely, we focus on linear models for seabed evolution and on methods which allow the removal of the explicit presence of the tide oscillations from them.

Some Models

for a small parameter ϵ and constants a, b and c, equation

$$\frac{\partial z^\epsilon}{\partial t} - \frac{a}{\epsilon}\nabla \cdot \Big((1 - b\epsilon\mathbf{m})g_a(|\mathbf{u}|)\nabla z^\epsilon\Big)$$

$$= \frac{c}{\epsilon}\nabla \cdot \left((1 - b\epsilon\mathbf{m})g_c(|\mathbf{u}|)\frac{\mathbf{u}}{|\mathbf{u}|}\right),$$

is a relevant model for the short-term dynamics of dunes. In the above equation, $z^\epsilon = z^\epsilon(x,t)$ where, for a given constant T, $t \in [0,T)$, stands for the dimensionless time and $x = (x_1, x_2) \in \mathbb{T}^2$, $\mathbb{T}^2$ being the two dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$, is the dimensionless position variable, is the dimensionless seabed altitude at time t and in position x. Operators ∇ and $\nabla\cdot$ refer to gradient and divergence. Functions g_a and g_c are regular on $\mathbb{R}^+$

Fields $\mathbf{u}$ and $\mathbf{m}$ are dimensionless water velocity and height. They are given by

$$\mathbf{u}(t,x) = \mathcal{U}(t, \frac{t}{\epsilon}, x) \quad \mathbf{m}(t,x) = \mathcal{M}(t, \frac{t}{\epsilon}, x), \quad (1)$$

where $\mathcal{U} = \mathcal{U}(t,\theta,x)$ and $\mathcal{M} = \mathcal{M}(t,\theta,x)$ are regular functions on $\mathbb{R}\times\mathbb{R}\times\mathbb{T}^2$,

$$\theta \longmapsto (\mathcal{U},\mathcal{M}) \text{ is periodic of period 1}$$

The following equation, for constants a, b and c

$$\frac{\partial z^\epsilon}{\partial t} - \frac{a}{\epsilon}\nabla\cdot\Big((1-b\sqrt{\epsilon}\mathbf{m})g_a(|\mathbf{u}|)\nabla z^\epsilon\Big)$$

$$= \frac{c}{\epsilon}\nabla\cdot\left((1-b\sqrt{\epsilon}\mathbf{m})g_c(|\mathbf{u}|)\frac{\mathbf{u}}{|\mathbf{u}|}\right),$$

with $\mathbf{u}$ and $\mathbf{m}$ given by

$$\mathbf{u}(t,x) = \widetilde{\mathcal{U}}(t,\frac{t}{\sqrt{\epsilon}},\frac{t}{\epsilon},x), \quad \mathbf{m}(t,x) = \mathcal{M}(t,\frac{t}{\sqrt{\epsilon}},\frac{t}{\epsilon},x), \tag{2}$$

is a relevant model for mean-term dune dynamics.

For mathematical reasons, we assume

$$\widetilde{\mathcal{U}}(t,\tau,\theta,x) = \mathcal{U}(t,\theta,x) + \sqrt{\epsilon}\,\mathcal{U}_1(t,\tau,\theta,x), \tag{3}$$

where $\mathcal{U} = \mathcal{U}(t, \theta, x)$ and $\mathcal{U}_1 = \mathcal{U}_1(t, \tau, \theta, x)$ are regular. We also assume that $\mathcal{M} = \mathcal{M}(t, \tau, \theta, x)$ is regular

A relevant model for long-term dune dynamics is the following equation

$$\frac{\partial z^\epsilon}{\partial t} - \frac{a}{\epsilon^2} \nabla \cdot ((1 - b\epsilon \mathbf{m}) g_a(\mathbf{u}) \nabla z^\epsilon)$$

$$= \frac{c}{\epsilon^2} \nabla \cdot \left((1 - b\epsilon \mathbf{m}) g_c(\mathbf{u}) \frac{\mathbf{u}}{|\mathbf{u}|} \right),$$

where a, b and c are constants, and where z is defined on the same space as before. It is also relevant to assume

$$\mathbf{u}(x, t) = \mathcal{U}(\frac{t}{\epsilon}, x) + \epsilon^2 \mathcal{U}_2(t, \frac{t}{\epsilon}, x),$$

$$\mathbf{m}(t, x) = \mathcal{M}(\frac{t}{\epsilon}, x) + \epsilon^2 \mathcal{M}_2(t, \frac{t}{\epsilon}, x) \qquad (4)$$

where $\mathcal{U} = \mathcal{U}(\theta, x)$, $\mathcal{U}_2(t, \theta, x)$, $\mathcal{M} = \mathcal{M}(\theta, x)$ and $\mathcal{M}_2 = \mathcal{M}_2(t, \theta, x)$ are regular

The above PDE equations need to be provided with an initial condition

$$z^{\epsilon}_{|t=0} = z_0, \tag{5}$$

giving the shape of the seabed at the initial time.

Modeling

Sand transport equation The equation modeling sand transport is the following (see Van Rijn, Idier):

$$\frac{\partial z}{\partial t} + \frac{1}{1-p}\nabla \cdot q = 0. \tag{6}$$

In this equation the fields depends on time $t \in [0, T)$, for $T > 0$, on the horizontal position $x = (x_1, x_2) \in \Omega$, where Ω is a regular open set of $\mathbb{R}^2$. The field $z = z(t, x)$ is the height of the seabed in position x and at time t and $q = q(x, t)$ is the sand volume flow in x and at t. The parameter $p \in [0, 1)$ is called sand porosity. Equation (6) has to be coupled with a low linking the sand flow q with the seabed height variation and the velocity of the water near the seabed. Usually, such a law is written

$$q = q_f - |q_f| \lambda \nabla z, \tag{7}$$

where q_f stands for the water velocity induced sand flow on a flat seabed and where $|q_f|$ stands for its norm. The constant λ is the inverse value of the maximum slope of the sediment surface when the water velocity is 0. A generic way to write q_f is

$$q_f = \alpha\, \widetilde{\chi}(g(|\mathbf{u}|) - g(u_c))\frac{\mathbf{u}}{|\mathbf{u}|}, \tag{8}$$

where g is a non-negative regular function defined on $\mathbb{R}^+$ and where $\widetilde{\chi}$ is a regular function from $\mathbb{R}$ to $\mathbb{R}$, being 0 on $\mathbb{R}^-$ and increasing on $\mathbb{R}^+$. $\mathbf{u}$ is the water velocity near the seabed, $g(u)$ is regular function of $u \in \mathbb{R}^+$ and u_c is the threshold under which the water velocity does

not make the sand move.
Every law encountered in the literature, for instance Meyer-Peter and Müller formula, Bagnold and Gadd formula and and Van Rijn formula, is recovered by setting functions χ and g.

In the sequel of the present paper we shall restrict ourselves to laws of the Van Rijn which consists in writting

$$q_f = \alpha \, \chi(D_G(|\tau_b| - \tau_c)) \, \frac{\tau_b}{|\tau_b|}, \tag{9}$$

where τ_b is the shear stress density imposed by the water on the seabed. It is linked with $\mathbf{u}$ by

$$\tau_b = \rho \frac{|\mathbf{u}|^2}{C^2} \frac{\mathbf{u}}{|\mathbf{u}|}, \tag{10}$$

where ρ is the water density, C is a constant defined by $C = \ln(\frac{12d}{3D_G})$, d being the water height above the seabed and D_G being the sand speck diameter. The threshold τ_c expresses as

$$\tau_c = \rho \frac{u_c^2}{C^2}, \tag{11}$$

and χ is given by

$$\begin{aligned} \chi(\sigma) &= 0 && \text{if } \sigma < 0, \\ &= |\sigma^{3/2}| && \text{if } \sigma \geq 0. \end{aligned} \tag{12}$$

The order of magnitude of constant α is 100.

Scaling

Now, we will scale (43) to write a dimensionless version of it. We introduce a characteristic time $\bar{t}$ and a characteristic length $\bar{L}$ and we define the dimensionless variables t' and x', making $\bar{t}$ and $\bar{L}$ the units by

$$t = \bar{t}t', \quad x = \bar{L}x'. \tag{13}$$

We also define $\bar{z}$ the characteristic height of the dunes and the dimensionless seabed height

$$z'(t', x') = \frac{1}{\bar{z}} z(\bar{t}t', \bar{L}x'). \tag{14}$$

Concerning coefficients of equation (43), we introduce $\bar{u}$ the characteristic velocity of the water, we consider the mean water height H and $\bar{M}$ the characteristic height variation due to the tide. Then we define u' being the dimensionless water height variation by

$$\mathbf{u}'(t',x') = \frac{1}{\bar{u}}\mathbf{u}(\bar{t}t',\bar{L}x'), \quad \mathbf{m}'(t',x') = \frac{1}{\bar{M}}(d(\bar{t}t',\bar{L}x')-H \tag{15}$$

Once those variables and fields are introduced, we first approximate C, taking into account that $\frac{\bar{M}}{H}$ is small.

$$C = \ln\left(\frac{4H}{D_G}\right)+\ln\left(1+\frac{\bar{M}}{H}\mathbf{m}'\right) \simeq \ln\left(\frac{4H}{D_G}\right)+\frac{\bar{M}}{H}\mathbf{m}'. \tag{16}$$

From (16) we get

$$\frac{1}{C^3} \simeq \frac{1}{\left(\ln\left(\frac{4H}{D_G}\right)\right)^3}\left(1-3\frac{\bar{M}}{H\ln\left(\frac{4H}{D_G}\right)}\mathbf{m}'\right). \tag{17}$$

Since for instance

$$\nabla z(\bar{t}t', \bar{L}x') = \frac{1}{\bar{z}\bar{L}} \nabla' z'(t', x'), \qquad (18)$$

we get from equation (43) the following equation for z':

$$\frac{\partial z'}{\partial t'} - \frac{\lambda}{1-p} \alpha \, \frac{\bar{t}\bar{u}^3 (\rho D_G)^{3/2}}{\left(\ln(\frac{4H}{D_G})\right)^3 \bar{L}^2} \, \nabla' \cdot$$

$$\left(\left(1 - 3 \frac{\bar{M}}{H \ln(\frac{4H}{D_G})} \mathbf{m}' \right) \chi \left(|\mathbf{u}'|^2 - \frac{{u_c}^2}{\bar{u}^2} \right) \nabla' z' \right)$$

$$= \frac{1}{1-p} \alpha \, \frac{\bar{t}\bar{u}^3 (\rho D_G)^{3/2}}{\left(\ln(\frac{4H}{D_G})\right)^3 \bar{L}\bar{z}} \, \nabla' \cdot \left(\left(1 - 3 \frac{\bar{M}}{H \ln(\frac{4H}{D_G})} \mathbf{m}' \right) \right. \qquad (19)$$

$$\left. \chi \left(|\mathbf{u}'|^2 - \frac{{u_c}^2}{\bar{u}^2} \right) \frac{\mathbf{u}'}{|\mathbf{u}'|} \right).$$

Having this dimensionless model on hand, we will now consider several situations in setting the characteristic values for short, mean and long-term dune evolution and for small and big sand specks.

First, we fix the characteristic sizes which are common for every situation. Dunes exist within coastal ocean waters over a relatively flat continental shelf, with a water height of about 30 to 50 meters, with tide induced height variations which are not too strong and with relatively strong tide currents. Then we set

$$\bar{u} = 1\, m/s, \qquad H = 50\, m, \qquad \bar{M} = 5\, m. \tag{20}$$

Moreover, the order of magnitude of coefficient $\frac{\lambda}{1-p}$ is 1, then we get

$$\frac{\lambda}{1-p} = 1 \text{ and } \frac{1}{1-p} = 2. \tag{21}$$

Now we detail the sizes of every characteristic value and of their concerned ratios in equation (17), (19) for every situation.

Short-term dynamics of dunes made of a small sand specks

Here, we shall consider that $\bar{t}$ is an observation period of time. We take as $\bar{t}$ the order of magnitude of the smallest period of time during which dunes undergo significant evolution in a tide-submitted environment, i.e. $\bar{t} = 100\,days \sim 2400\,hours \sim 8.6\ 10^6\,s$. Introducing $\bar{\omega}$ the main tide frequency, $\bar{t}$ has to be compared with the main tide period $\frac{1}{\bar{\omega}} \sim 13\,hours \sim 4.7\ 10^4\,s$. This leads to the definition of a small parameter ϵ:

$$\frac{1}{\bar{t}\bar{\omega}} \sim \frac{1}{200} = \epsilon. \tag{22}$$

We consider that the sand speck diameter D_G is $0.1mm = 10^{-4}\,m$. According to Flemming and Idier, this gives rise to dunes being about 1 meter high, the wave length of which is about 10 meters. Then we set

$$\bar{z} = 1\,m \text{ and } \bar{L} = 10\,m. \tag{23}$$

We also consider that the critical velocity u_c is small compared with $\bar{u}$. In other words we set

$$\frac{u_c^2}{\bar{u}^2} = 0. \tag{24}$$

As the computations of the factors in (23) yields

$$\begin{gathered}
\frac{\lambda}{1-p}\alpha\frac{\bar{t}\bar{u}^3(\rho D_G)^{3/2}}{\left(\ln(\frac{4H}{D_G})\right)^3\bar{L}^2} \sim 90 \sim \frac{1}{2\epsilon}, \\
\frac{\lambda}{1-p}\alpha\frac{\bar{t}\bar{u}^3(\rho D_G)^{3/2}}{\left(\ln(\frac{4H}{D_G})\right)^3\bar{L}\bar{z}} \sim 1800 \sim \frac{10}{\epsilon}, \\
\frac{3\bar{M}}{H\ln(\frac{4H}{D_G})} \sim 2.10^{-2} \sim 4\epsilon,
\end{gathered} \tag{25}$$

equation (19) reads

$$\frac{\partial z}{\partial t} - \frac{1}{2\epsilon}\nabla\cdot((1-4\epsilon\mathbf{m})|\mathbf{u}|^3\nabla z) = \frac{10}{\epsilon}\nabla\cdot((1-4\epsilon\mathbf{m})|\mathbf{u}|^2\mathbf{u}), \tag{26}$$

where we removed the '.

Concerning fluid fields $\mathbf{u}$ and $\mathbf{m}$, we assume that they are periodic functions, with modulated amplitude, and of period the tide period. In other words

$$\mathbf{u}(x,t) = \mathcal{U}(t, \frac{t}{\epsilon}, x), \quad \mathbf{m}(x,t) = \mathcal{M}(t, \frac{t}{\epsilon}, x), \tag{27}$$

for functions $\mathcal{U}$ and $\mathcal{M}$ being regular, and such that $\theta \longmapsto (\mathcal{U}(t,\theta,x), \mathcal{M}(t,\theta,x))$ is periodic of period 1, with a null mean value.

Finally, as dunes of the considered kind are, in nature, gathered into dunes fields it is not completely unrealistic to set equation (26) in a periodic position space.

As matter of the fact, considering equation (26) is appropriate for the study of short-term dynamics of dunes made of small sand specks with a mathematical point of view.

Short-term dynamics of dunes made of a big sand specks

For this regime, we consider:

$$\bar{t} \sim 100\,days \sim 2400\,hours \sim 8.6\;10^6\,s,$$
$$\frac{1}{\bar{\omega}} \sim 13\,hours \sim 4.7\;10^4\,s \qquad D_G = 5\;10^{-3}\,m$$
$$\bar{z} = 50\,m,\;\; \bar{L} = 300\,m,\;\; u_c = \frac{1}{2}\,m/s. \tag{28}$$

Then

$$\begin{aligned}
&\frac{\lambda}{1-p}\alpha\frac{\bar{t}\bar{u}^3(\rho D_G)^{3/2}}{\left(\ln(\frac{4H}{D_G})\right)^3\bar{L}^2} \sim 90 \sim \frac{1}{2\epsilon},\\
&\frac{\lambda}{1-p}\alpha\frac{\bar{t}\bar{u}^3(\rho D_G)^{3/2}}{\left(\ln(\frac{4H}{D_G})\right)^3\bar{L}\bar{z}} \sim 1000 \sim \frac{5}{\epsilon},\\
&\frac{3\bar{M}}{H\ln(\frac{4H}{D_G})} \sim 1.3\;10^{-2} \sim 3\epsilon,
\end{aligned} \tag{29}$$

equation (19), with the ' removed, gives

$$\frac{\partial z}{\partial t} - \frac{1}{2\epsilon}\nabla\cdot\left((1-3\epsilon\mathbf{m})\chi(|\mathbf{u}|^2-\frac{1}{2})\nabla z\right)$$

$$= \frac{5}{\epsilon}\nabla \cdot \left((1 - 3\epsilon\mathbf{m})\chi(|\mathbf{u}|^2 - \frac{1}{2})\frac{\mathbf{u}}{|\mathbf{u}|} \right).$$

Mean-term dynamics of dunes made of a small sand specks

By mean-term we mean a period of time of $4.5\,years \sim 54\,months \sim 1.4\ 10^8\,s$. Then, we take $\bar{t} = 1.4\ 10^8\,s$, which is compared with $\frac{1}{\bar{\omega}} \sim 13\,hours \sim 4.7\ 10^4\,s$ giving

$$\frac{1}{\bar{t}\bar{\omega}} \sim \frac{1}{3000} = \epsilon. \tag{30}$$

We also consider a second tide period which is the time for the earth, the moon and the sun to recover approximately the same relative positions. This period of time $\frac{1}{\bar{\omega}_c}$ is about one month. So we have

$$\frac{1}{\bar{t}\bar{\omega}_c} \sim \frac{1}{54} \sim \sqrt{\epsilon}. \tag{31}$$

We also take $D_G = 5\ 10^{-5}\,m$ and

$$\bar{z} = 1\,m,\ \bar{L} = 10\,m,\ u_c = 0. \tag{32}$$

Computing the coefficients in equation (19) gives

$$\frac{\partial z}{\partial t}-\frac{1}{\epsilon}\nabla\cdot((1-\sqrt{\epsilon}\,\mathbf{m})|\mathbf{u}|^3\,\nabla z)=\frac{20}{\epsilon}\nabla\cdot((1-\sqrt{\epsilon}\,\mathbf{m})|\mathbf{u}|^2\mathbf{u}). \tag{33}$$

As was previously seen, it is reasonable to set this equation in a periodic domain and concerning the fluid fields we consider

$$\mathbf{u}(t,x)=\widetilde{\mathcal{U}}(t,\frac{t}{\sqrt{\epsilon}},\frac{t}{\epsilon}),\ \mathbf{m}(t,x)=\mathcal{M}(t,\frac{t}{\sqrt{\epsilon}},\frac{t}{\epsilon}). \tag{34}$$

to take into account the two tide periods under consideration. In (34) we take $\mathcal{U}$ and $\mathcal{M}$ as regular functions such that

$$\begin{cases} \tau \longmapsto (\widetilde{\mathcal{U}}(t,\tau,\theta,x),\mathcal{M}(t,\tau,\theta,x)) \\ \theta \longmapsto (\widetilde{\mathcal{U}}(t,\tau,\theta,x),\mathcal{M}(t,\tau,\theta,x)) \end{cases} \tag{35}$$

are periodic of period 1.

Long-term dynamics of dunes made of small sand specks

We take here $\overline{t} \sim 16\,years \sim 1.4\ 10^5\,hours \sim$

$5\ 10^9 s$. We compare this period of time with the second tide period $\frac{1}{\bar{\omega}_c} \sim 1\, month \sim 2.6\ 10^6 s$. Then, we define ϵ by

$$\frac{1}{\bar{t}\bar{\omega}_c} \sim \frac{1}{192} = \epsilon. \tag{36}$$

We set

$$D_G = 7.10^{-5}\, m,\ \bar{z} = 1\, m,\ \bar{L} = 10\, m,\ u_c = 0\, m/s. \tag{37}$$

with those values equation (19) yields

$$\frac{\partial z}{\partial t} + \frac{1}{\epsilon^2} \nabla \cdot ((1 - 4\epsilon \mathbf{m}) |\mathbf{u}|^3 \nabla z) = \frac{20}{\epsilon^2} \nabla \cdot ((1 - 4\epsilon \mathbf{m}) |\mathbf{u}|^2 \mathbf{u}). \tag{38}$$

As, at the second tide period scale the tide phenomena may almost be considered as really periodic we set

$$\begin{aligned} \mathbf{u}(x,t) &= \mathcal{U}(\frac{t}{\epsilon}, x) + \epsilon^2 \mathcal{U}_2(t, \frac{t}{\epsilon}, x), \\ \mathbf{m}(x,t) &= \mathcal{M}(\frac{t}{\epsilon}, x) + \epsilon^2 \mathcal{M}_2(t, \frac{t}{\epsilon}, x), \end{aligned} \tag{39}$$

where $\mathcal{U}$, $\mathcal{U}_2$, $\mathcal{M}$ $\mathcal{M}_2$ are regular functions such that $\theta \longmapsto (\mathcal{U}(\theta,x), \mathcal{U}_2(t,\theta,x), \mathcal{M}(\theta,x), \mathcal{M}_2(t,\theta,x))$ is periodic of period 1 and such that

$$\int_0^1 \mathcal{U}(\theta,x)d\theta = 0, \tag{40}$$

$$\int_0^1 \mathcal{M}(\theta,x)d\theta = 0. \tag{41}$$

Injecting equation (10) into (9) and (7) we get

$$q = \alpha\,\chi\left(D_G\,\rho\,\frac{|\mathbf{u}|^2 - {u_c}^2}{C^2}\right)\left(\frac{\mathbf{u}}{|\mathbf{u}|} - \lambda\nabla z\right), \tag{42}$$

and equation (6) reads

$$\frac{\partial z}{\partial t} + \frac{\alpha}{1-p}\nabla\cdot\left[\chi\left(D_G\,\rho\,\frac{|\mathbf{u}|^2 - {u_c}^2}{C^2}\right)\left(\frac{\mathbf{u}}{|\mathbf{u}|} - \lambda\nabla z\right)\right] = 0. \tag{43}$$

Homogenization:Main Results

The first concerns existence and uniqueness for the short and mean-term models.

Under technical assumptions, we have

Theorem

For any $T > 0$, any $a > 0$, any real constants b and c and any $\epsilon > 0$, (under additional assumptions, see the paper) if

$$z_0 \in L^2(\mathbb{T}^2), \tag{44}$$

there exists a unique function $z^\epsilon \in L^\infty([0,T), L^2(\mathbb{T}^2))$ solution to equation (**??**) or (**??**) provided with initial condition (5).

Moreover, for any $t \in [0,T]$, z^ϵ satisfy

$$\|z^\epsilon\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \leq \widetilde{\gamma}, \tag{45}$$

for a constant $\widetilde{\gamma}$ not depending on ϵ and

$$\frac{d\left(\int_{\mathbb{T}^2} z^\epsilon(t,x)\,dx\right)}{dt} = 0. \tag{46}$$

Remark

As equations are linear, almost parabolic equations, the proof of the existence of z^ϵ over a time interval depending on ϵ is a straight forward consequence of adaptations of results from Ladyzenskaja, Solonnikov and Ural'Ceva or J.L.Lions. But here, since we want to follow an asymptotic process consisting in making $\epsilon \to 0$, we need a time interval which does not depend on ϵ. Because of the presence of $\frac{1}{\epsilon}$ factor and the fact that the diffusion term may cancel, the proof of theorem needs several steps. In a first step, we prove the existence of a solution, periodic in time and space of a parabolic equation. From this first existence result, we deduce existence of a solution, periodic in time and space of an ad-doc degenerate parabolic equation.

Those two results are interesting by themselves and complete the theorem collection in the topic of time and space time-periodic solution to parabolic equation. Concerning this topic, we refer for instance to Barles and Souganidis, Berestycki, Hamel and Roques, Bostan, Hansbo, Kono, Nadin, Namah and Roquejoffre Pardoux, Da Lio, Norris, Park and Tanabe, Pardoux, Petita and Tanabe.

We now give a result concerning the asymptotic behavior as $\epsilon \longrightarrow 0$ of the short-term model.

Theorem

For any $T > 0$, under the same assumptions as in the above theorem, the solution z^ϵ to equation (short or mean term) given by the above theorem two-scale converges to a profile $U \in L^\infty([0,T], L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2)))$ which is the unique solution to

$$\frac{\partial U}{\partial \theta} - \nabla \cdot (\tilde{\mathcal{A}} \nabla U) = \nabla \cdot \tilde{\mathcal{C}}, \tag{47}$$

where $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{C}}$ are given by

$$\widetilde{\mathcal{A}} = a\, g_a(|\mathcal{U}(t,\theta,x)|) \text{ and } \widetilde{\mathcal{C}} = c\, g_c(|\mathcal{U}(t,\theta,x)|)\,\frac{\mathcal{U}(t,\theta,x)}{\mathcal{U}(t,\theta,x)} \quad (48)$$

In this theorem, $L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))$ stands for the space functions depending on θ and x mapping $\mathbb{R}$ to $L^2(\mathbb{T}^2)$ and which are periodic of period 1 with respect to θ and $L^\infty([0,T], L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2)))$ stands for the space of functions mapping $[0,T]$ to $L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))$ and which are bounded. For the definition and results about two-scale convergence we refer to Nguetseng, Allaire and Frénod Raviart and Sonnendrücker.

Finally, we give a corrector result for the short-term model under restrictive assumptions.

Theorem

Under the same assumptions as in the above theorem and if moreover $U_{thr} = 0$, considering function $z^\epsilon \in L^\infty([0,T), L^2(\mathbb{T}^2))$, solution to short term equation with initial condition and function $U^\epsilon \in L^\infty([0,T], L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2)))$ defined by

$$U^\epsilon(t,x) = U(t, \frac{t}{\epsilon}, x), \tag{49}$$

where U is the solution to (47), the following estimate is satisfied:

$$\left\| \frac{z^\epsilon - U^\epsilon}{\epsilon} \right\|_{L^\infty([0,T), L^2(\mathbb{T}^2))} \leq \alpha, \tag{50}$$

where α is a constant not depending on ϵ. Furthermore

$$\frac{z^\epsilon - U^\epsilon}{\epsilon}$$

two-scale converges to a profile

$$U^1 \in L^\infty([0,T], L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))),$$

which is the unique solution to

$$\frac{\partial U^1}{\partial \theta} - \nabla \cdot \left(\tilde{\mathcal{A}} \nabla U^1 \right) = \nabla \cdot \tilde{\mathcal{C}}_1 + \frac{\partial U}{\partial t} + \nabla \cdot (\tilde{\mathcal{A}}_1 \nabla U), \tag{51}$$

where $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{C}}$ are given by (48) and where $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{C}}_1$ are given by

$$\widetilde{\mathcal{A}}_1(t,\theta,x) = -ab\mathcal{M}(t,\theta,x)\, g_a(|\mathcal{U}(t,\theta,x)|)$$

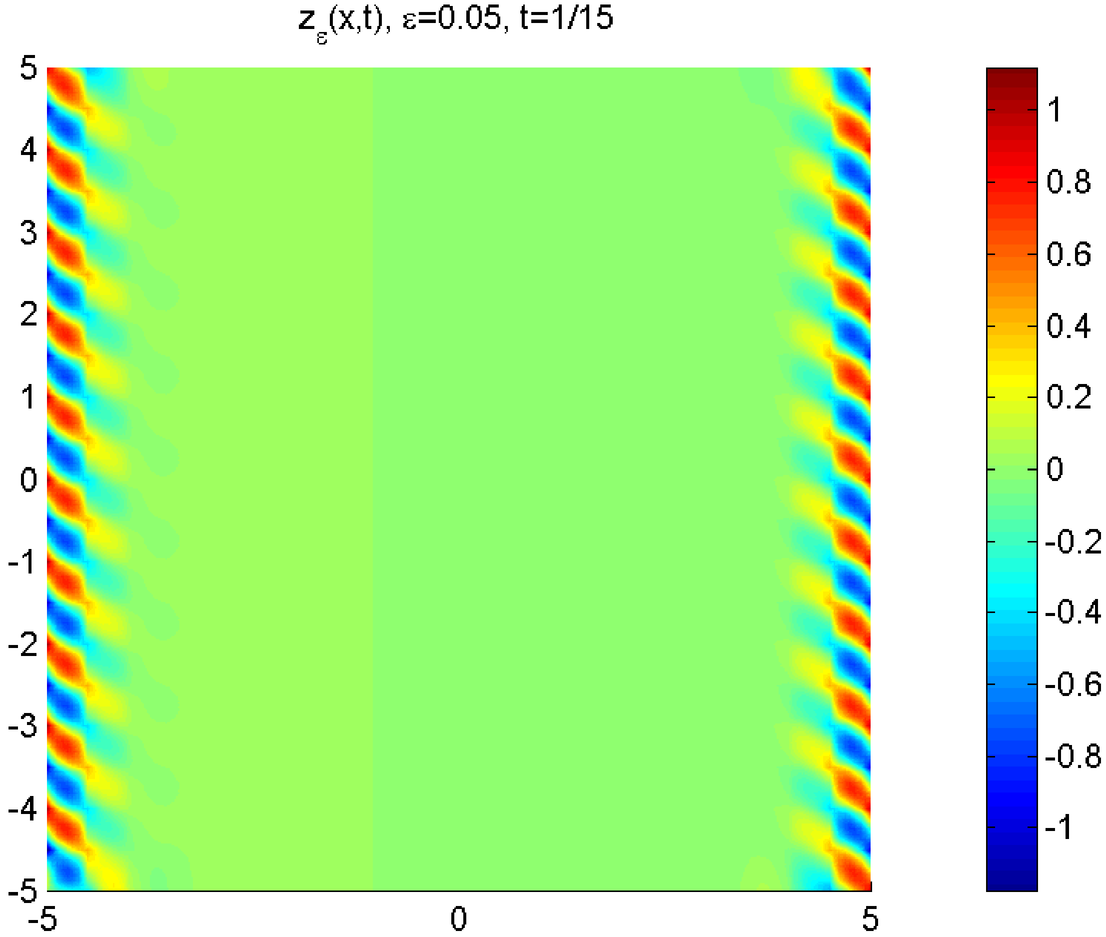
$$\widetilde{\mathcal{C}}_1(t,\theta,x) = -cb\mathcal{M}(t,\theta,x)\, g_c(|\mathcal{U}(t,\theta,x)|)\, \frac{\mathcal{U}(t,\theta,x)}{|U(t,\theta,x)|}.$$

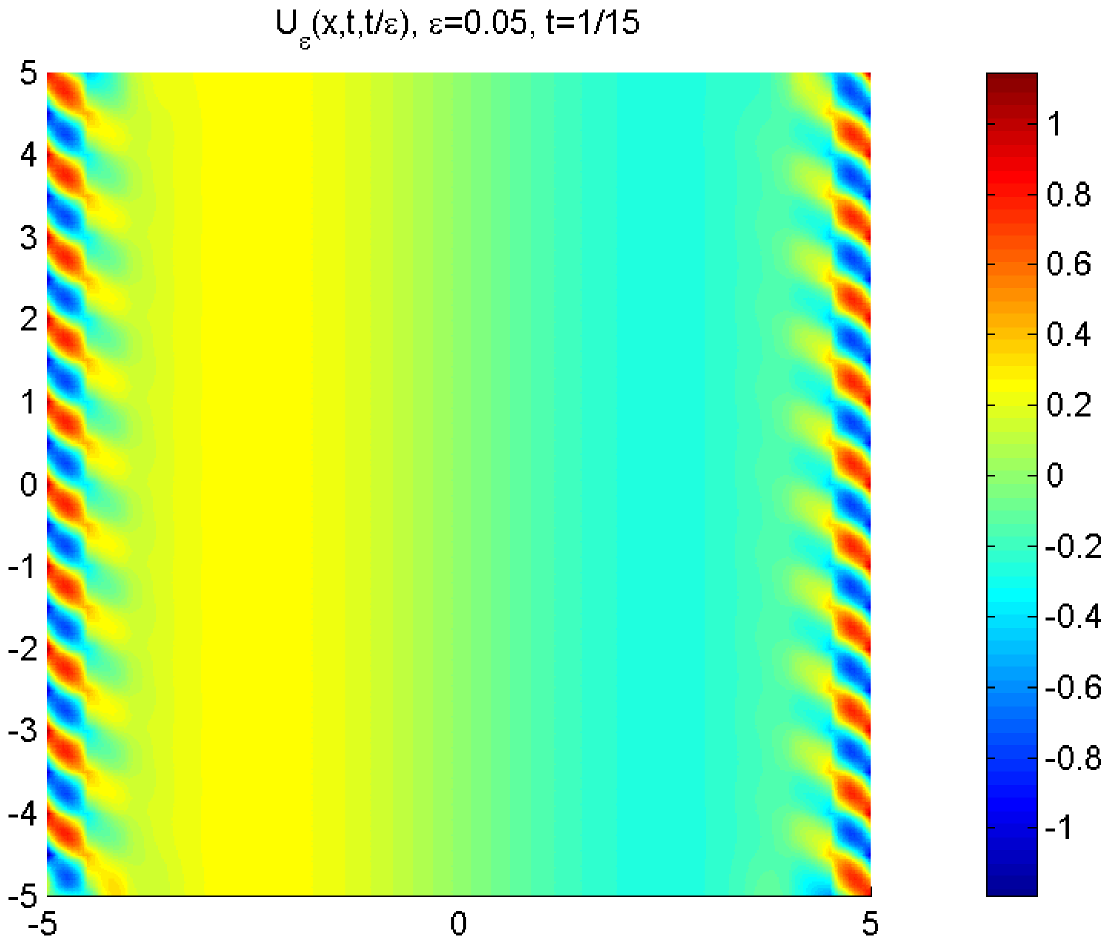
Remark

These two theorems state a rigorous version of asymptotic expansion of z^ϵ:

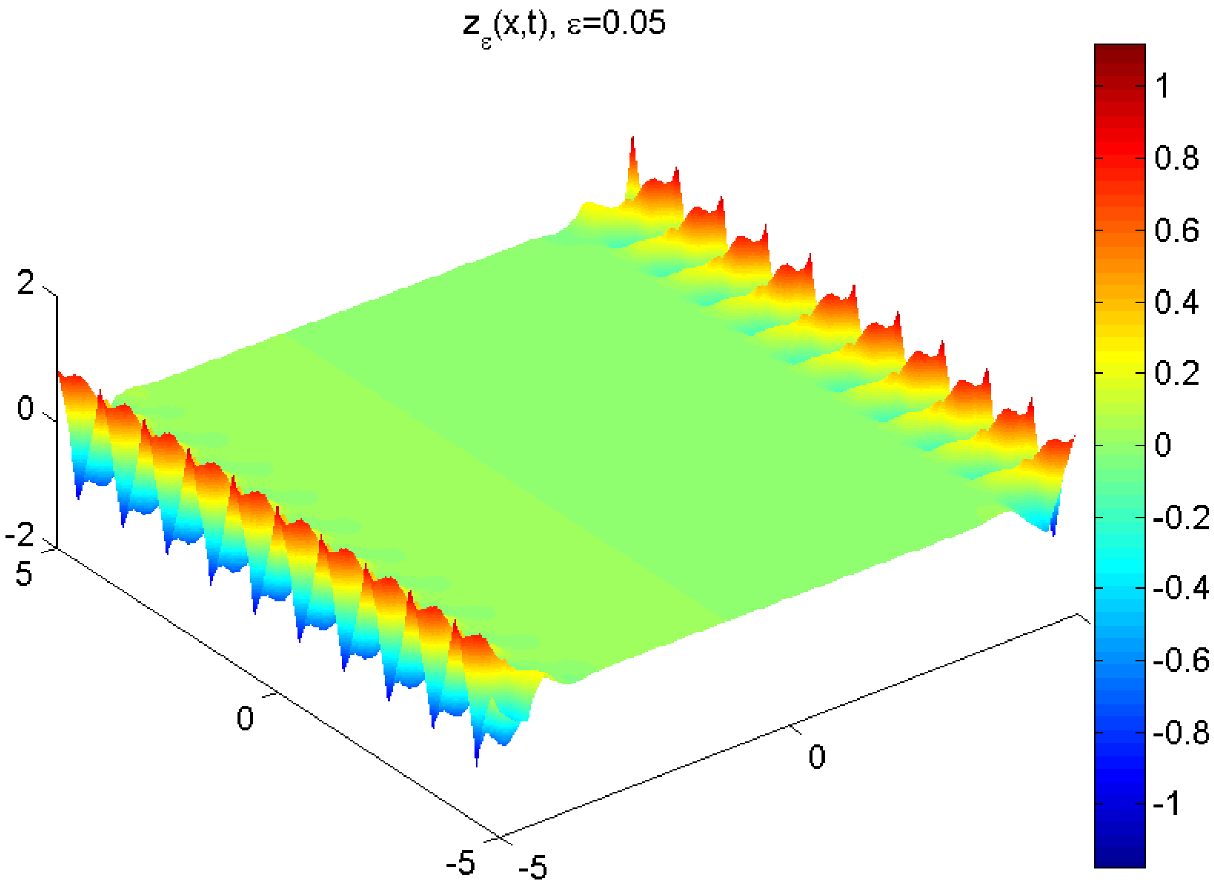
$$z^\epsilon(t,x) = U(t,\frac{t}{\epsilon},x) + \epsilon U1(t,\frac{t}{\epsilon},x) + \dots \quad (52)$$

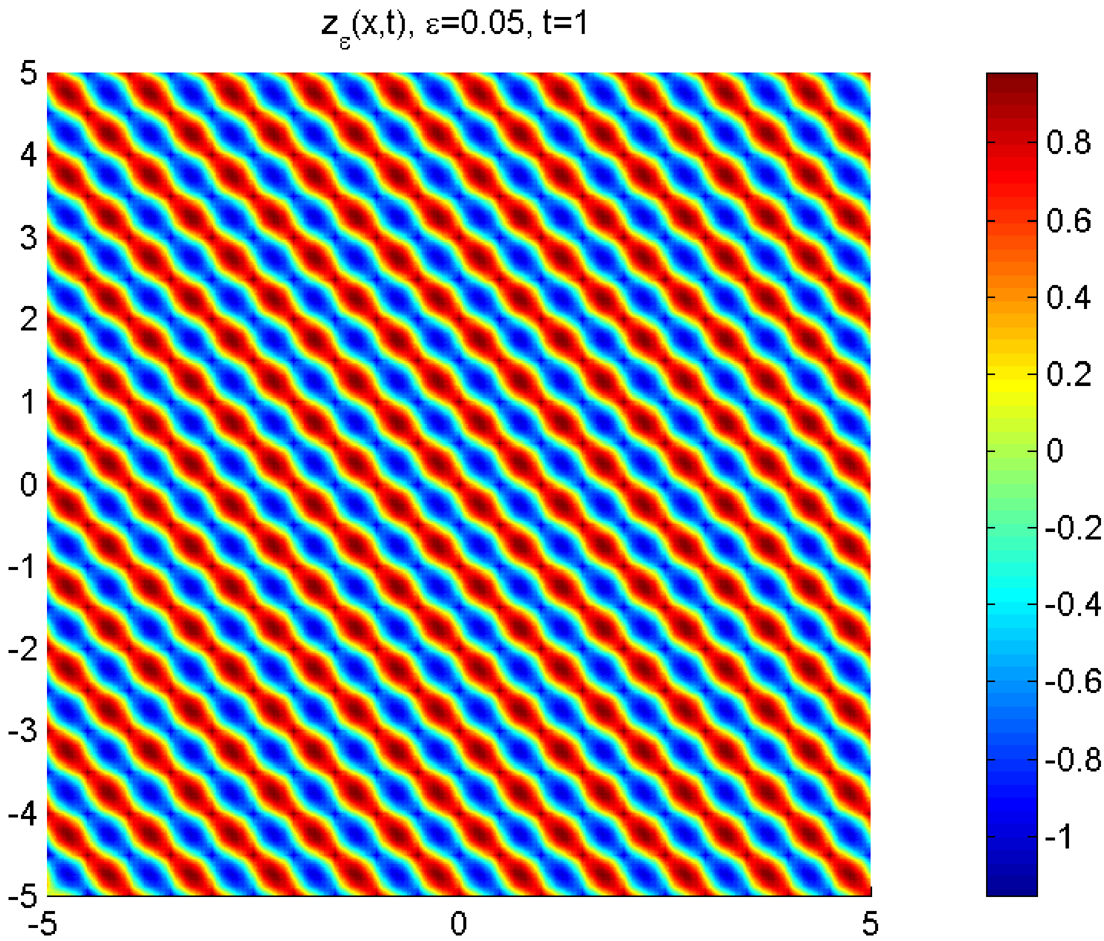
Numerical simulations, Getfem++



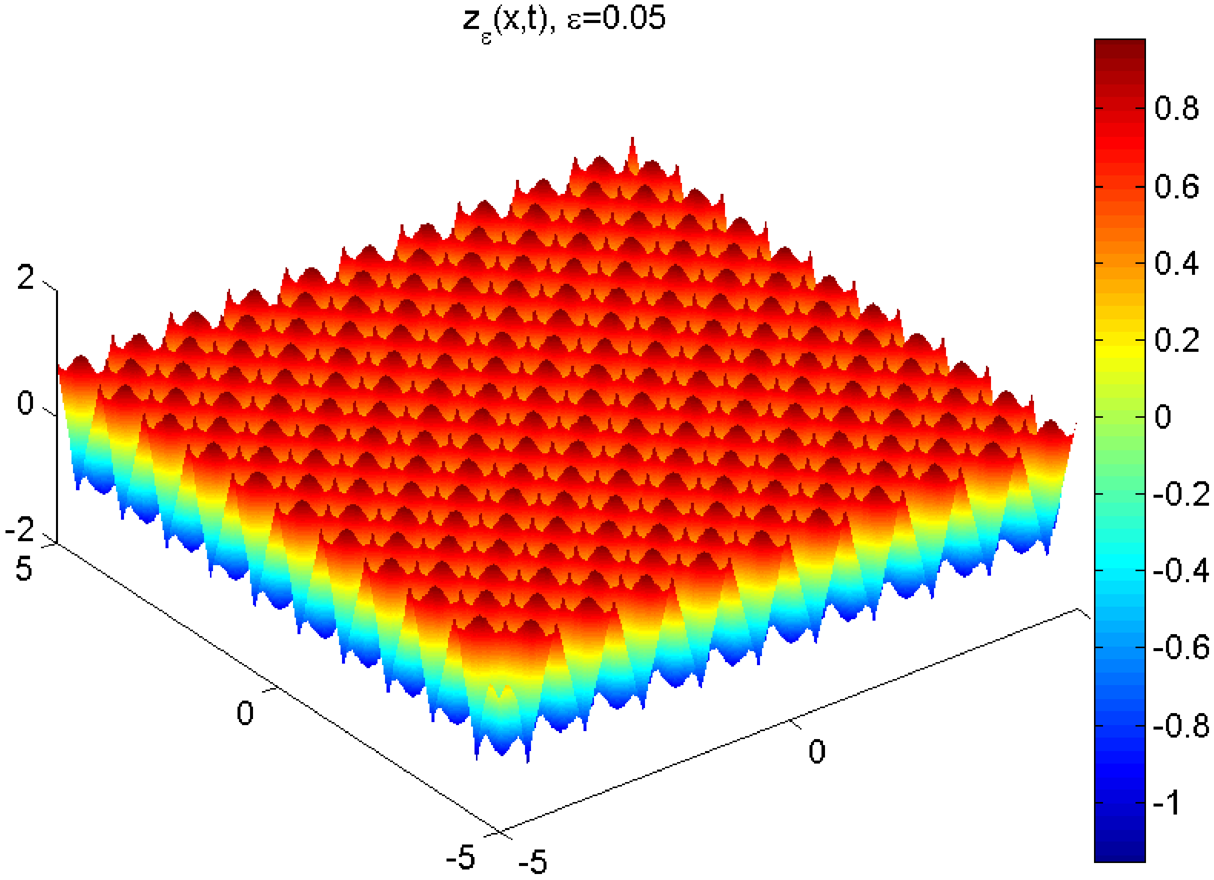


$U_\varepsilon(x,t,t/\varepsilon)$, ε=0.05, t=1/15
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
0
5
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1


$z_\varepsilon(x,t)$, ε=0.05
2
0
-2
5
0
-5
-5
0
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1



$z_\varepsilon(x,t)$, ε=0.05, t=1
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
0
5
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1



z_ε(x,t), ε=0.05
2
0
-2
5
0
-5
-5
0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1

Perspectives

1)The case where $\mathbf{u}$ satisfies incompressible Navier Stokes equations in some cases

2) Long term case: I. Faye, E. Frénod, D. Seck, almost finished

3) Numericals aspects to be improved: I. Faye, E. Frénod, D. Seck

4) Consideration of the swell.

5) Problems of optimal mass transportation, of controllability, of shape and topological optimization

II: Pollution problems

Topological optimization approach

Joint works with I. FAYE(University of Bambey in Sénégal) and A. SY (University of Bambey in Sénégal) Mathematical Modeling, Simulation, Visualization and e-Learning, Springer-Verlag Berlin Heidelberg 2008, pp 209-237

Modelling

The equations of mass conservation are given by

$$\begin{cases} \frac{\partial(\rho_s \varepsilon)}{\partial t} + div(\rho_s \varepsilon V) = 0 \text{ in } \Omega \\ \frac{\partial(\varepsilon \rho_s W)}{\partial t} + div(W \rho_s q + J) = 0 \text{ in } \Omega \end{cases} \tag{53}$$

and the equations of conservation of the momentum

$$\begin{cases} q = -\frac{K}{\mu}(\nabla p + \rho_s g e_3) \text{ Darcy's law} \\ J = -\rho_s D \nabla W \text{ Fick' s law} \end{cases}$$

where ρ_s in the unsaturated case is given by

$$\rho_s = \rho_0 \exp(\beta_T(T - T_0) + \beta_p(p - p_0) + \gamma W)$$

Replacing q and J by their expressions in (53) and suppose that

H-1 ρ_s is a constant. **H-2** The hydraulic conductivity tensor is a constant positive: $(\frac{K}{\mu} = \beta I_3, \beta > 0)$ and D is a constant positive $(D = aI_3, a > 0)$ and I_3 is the identity matrix of order 3.

$$\left\{ \begin{array}{c} \frac{\partial \varepsilon}{\partial t} - \beta \Delta p = 0 \quad \Omega \times (0, T_1) \\ \varepsilon(x, 0) = \varepsilon_0, \, \Omega \times \{t = 0\} \\ \varepsilon = \varepsilon_1 \, \partial\Omega \setminus \Gamma_1 \times (0, T_1) \\ \varepsilon = \varepsilon_s \, \Gamma_1 \times (0, T_1) \end{array} \right. \tag{54}$$

and

$$\left\{ \begin{array}{c} \varepsilon \frac{\partial W}{\partial t} - \frac{k}{\mu} \nabla W \nabla p - \frac{k}{\mu} \rho_s g \frac{\partial W}{\partial z} - a \Delta W = 0, \, \Omega \times (0, T_1) \\ \frac{\partial W}{\partial n} = 0 \, \partial\Omega \backslash \Gamma_1 \times (0, T_1) \\ W = V \, \Gamma_1 \times (0, T_1) \\ W(x, 0) = W_0, \; \Omega \times \{t = 0\} \end{array} \right. \tag{55}$$

H-3 In the porous medium we have a steady state, this means that $\frac{\partial}{\partial t} = 0$.

H-4 The evolution is isotherm.
Using hypothesis (**H-1**) and (**H-4**) we can find a relation between p the pressure and W the concentration:

$$\nabla p = -\frac{\gamma}{\beta_p}\nabla W.$$

Finally we obtain (by using the Van Genuchten law, (1980)):

$$\begin{cases} -\Delta p \;=\; 0 \,\text{dans}\, \Omega \\ p = \frac{[(\frac{\varepsilon_1 - \varepsilon_r}{\varepsilon_s - \varepsilon_r})^{-\frac{1}{m}} - 1]^{\frac{1}{n}}}{\alpha} \,\text{on}\, \partial\Omega \setminus \Gamma_1 \\ p = 0 \,\text{on}\, \Gamma_1 \end{cases} \tag{56}$$

$$\begin{cases} +\beta|\nabla W|^2 - \beta\rho_s g\frac{\partial W}{\partial z} - \frac{D_0}{\rho_0}\Delta W = 0 \,\text{in}\, \Omega \\ \frac{\partial}{\partial n}W = 0 \,\text{in}\, \partial\Omega\setminus\Gamma_1 \\ W = V \,\text{in}\, \Gamma_1 \end{cases} \tag{57}$$

Study of PDE
Fixed point theorem

Fixed point approach

The problem is to solve the following

$$\begin{cases} -\alpha\Delta u + \beta|\nabla u|^2 - \gamma\frac{\partial u}{\partial x_N} & = & 0 \text{ in } \Omega \\ u & = & 0 \text{ on } \Gamma^1 \\ \frac{\partial u}{\partial n} & = & 0 \text{ on } \Gamma^2 \end{cases} \quad (58)$$

Proposition

Let Ω be a bounded open set of $\mathbb{R}^N$. The above boundary value problem admits a solution

$$u \in K = \{v \in H_0^1(\Omega) \text{ tel que } m < div(\nabla v) < 0\}$$

where m is a negative constant.

Change of variables approach

We consider here an initial-value problem for a quasi-linear parabolic equation:

$$\begin{cases} u_t - \Delta u + \alpha|\nabla u|^2 + \frac{\partial u}{\partial z} & = & 0 & \text{in} & \Omega \times (0, T_1) \\ u & = & f & \text{on} & \partial\Omega \times (0, T_1) \\ u(x, 0) & = & g & \text{in} & \Omega \times (t = 0) \end{cases} \quad (59)$$

where $\alpha > 0$.

Proposition

The problem (59) has at least one solution.

Hints of proof Let us suppose at first that u is the solution of (59). Let us set

$$\omega := \phi(u),$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth function, not yet specified. We will choose ϕ such that ω solve a linear equation. We have

$$\omega_t = \phi'(u) u_t, \ \nabla \omega = \phi'(u) \nabla u,$$

and

$$\Delta \omega = \phi'(u) \Delta u + \phi''(u) |\nabla u|^2$$

and consequently (59) implies

$$\omega_t = \phi'(u) u_t = \phi'(u) \left[\Delta u - \alpha |\nabla u|^2 - \frac{\partial u}{\partial z} \right]$$

$$= \underbrace{\phi'(u) \Delta u}_{= \Delta \omega - \phi''(u) |\nabla u|^2} - \alpha \phi'(u) |\nabla u|^2 - \underbrace{\phi'(u) \frac{\partial u}{\partial z}}_{= \frac{\partial \omega}{\partial z}}$$

$$\omega_t = \Delta\omega - |\nabla u|^2(\phi''(u) + \alpha\phi'(u)) - \frac{\partial\omega}{\partial z}$$

Thus

$$\omega_t - \Delta\omega + \frac{\partial\omega}{\partial z} = -|\nabla u|^2(\phi''(u) + \alpha\phi'(u)) \ = \ 0$$

provided that we choose ϕ to satisfy $\phi''(u) + \alpha\phi'(u) = 0$.

Topological Sensitivity We optimize the functional

$$J(\Omega, W) = \int_\Omega (W - W_1)^2 dx$$

where W_1 is a given function in $L^2(\Omega)$ which represents a target and W is the solution of the partial differential equation

$$\left\{ \begin{array}{c} +\beta|\nabla W|^2 - \beta\rho_s g\frac{\partial W}{\partial z} - a\Delta W = 0 \quad \text{in} \quad \Omega \\ \frac{\partial}{\partial n}W = 0 \quad \text{on} \quad \partial\Omega\backslash\Gamma^1 \\ W = V \quad \text{on} \quad \Gamma^1 \end{array} \right. \tag{60}$$

with α, β and γ are positive constants
The porosity of the medium also is given by the Van Genuchten law

$$\varepsilon = (\varepsilon_s - \varepsilon_r)(1 + (\alpha p)^n)^{-m} + \varepsilon_r.$$

In fact using the relation between p and W the topological optimization problem in W is equivalent to look for Ω with

$$\min_{\Omega(\epsilon)} J(\Omega(\epsilon))$$

where

$$J(\Omega(\epsilon)) = \int_{\Omega(\epsilon)} (1/\gamma[\ln \frac{\rho_s}{\rho_0} - \beta_p(p_\epsilon - p_0)] - W_1)^2 dx \tag{61}$$

where: $\epsilon \in (0, 1)$ is a small parameter and p_ϵ is solution of

$$\begin{cases} -\Delta p_\epsilon = 0 & \text{in} \quad \Omega(\epsilon) \\ p_\epsilon = p_1 & \text{on} \quad \partial\Omega \setminus \Gamma_1 \\ p_\epsilon = 0 & \text{on} \quad \Gamma_1 \\ p_\epsilon = 0 & \text{on} \quad \partial\omega_\epsilon \end{cases} \tag{62}$$

Theorem

Let

$$J_\Omega(p) = \int_\Omega \left| \frac{1}{\gamma} \left(\ln\left(\frac{\rho_s}{\rho_0}\right) - \beta_p(p - p_0) \right) - W_1 \right|^2 dx$$

be the cost function. Let $V \in \mathcal{V}_R$ be the solution to the adjoint equation

$$a_0(V, w) = -DJ(p, w) \quad \forall\, w \in \mathcal{V}_R. \qquad (63)$$

Then the function $j(\epsilon) = J_{\Omega_\epsilon}(p_\epsilon)$ have the following asymptotic expansion.

$$j(\epsilon) = j(0) + f(\epsilon)\delta_a(p, V) + \delta_J(p) - \delta_l(V) + o(f(\epsilon)) \qquad (64)$$

The function $\delta_j(x_0) = \delta_a(p(x_0), V(x_0)) + \delta_J(p(x_0)) - \delta_l(V(x_0))$ is called topological sensitivity or topological gradient and can be used as descent direction in optimization processus. Moreover, as j is independent of R and δ_j is independent of ϵ, it follows from the uniqueness of the asymptotic expansion that δ_j is also independent of R.

When $\omega_\epsilon = B(x_0, \epsilon)$ is a ball, δ_a, δ_J can be computed explicitly and we have:

$$\delta j(x_0) = -4\pi(p(x_0).V(x_0))$$

where p is the solution of the direct state V is the solution of the adjoint equation.

Remark

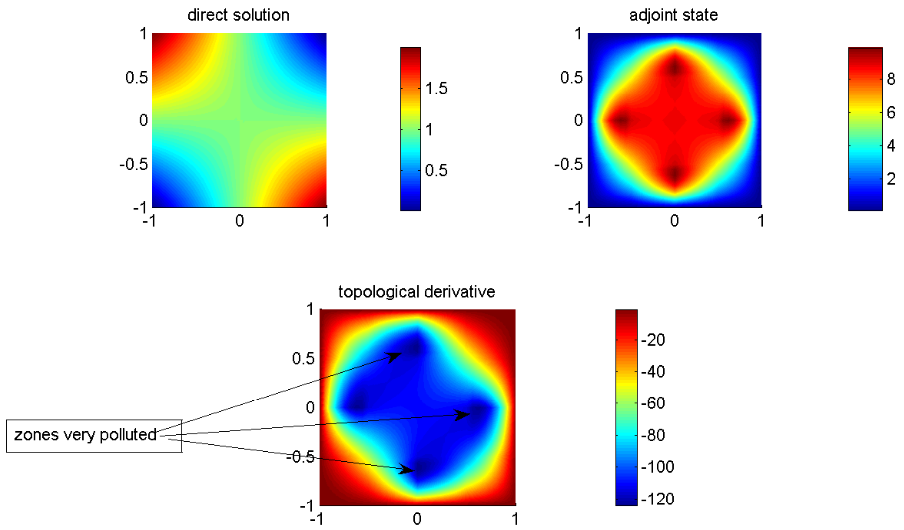
In the case of Neumann condition on the boundary of the hole, the topological sensitivity can be computed. When $\omega_\epsilon = B(x_0, \epsilon)$, we have:

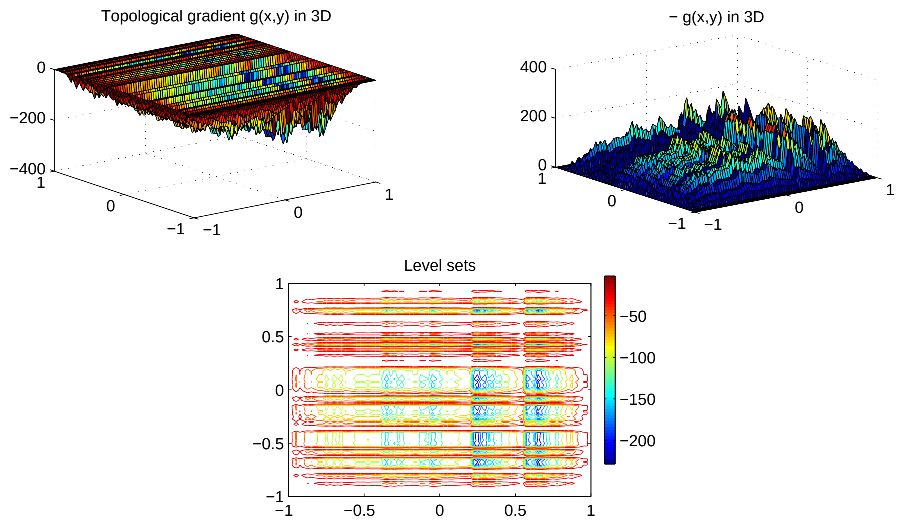
$$\delta j(x_0) = -4\pi \left(\nabla p(x_0).\nabla V(x_0) + \left| \frac{1}{\gamma} \left[\ln(\frac{\rho_s}{\rho_0}) - \beta_p(p(x_0) - p_0(x_0)) \right] - W_1(x_0) \right|^2 \right)$$

where p is the solution of the direct state V is the solution of the adjoint equation.

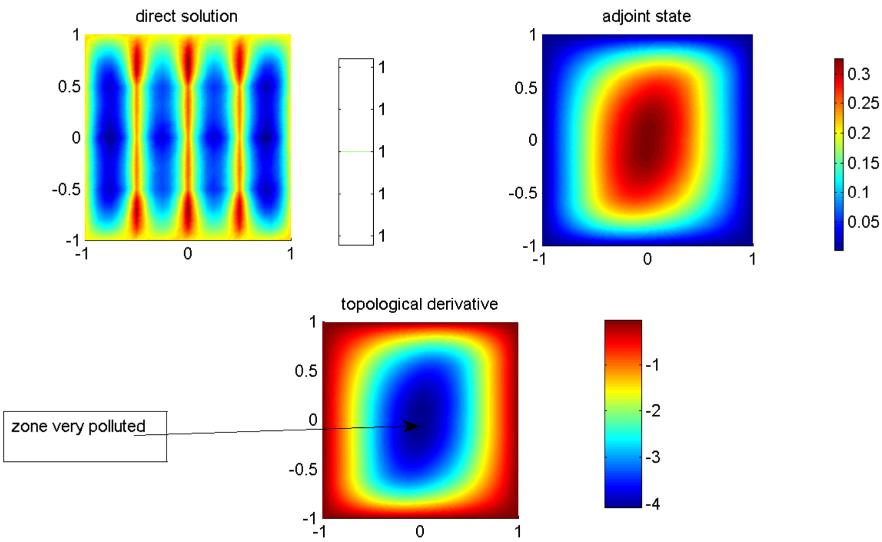
Numerical simulations

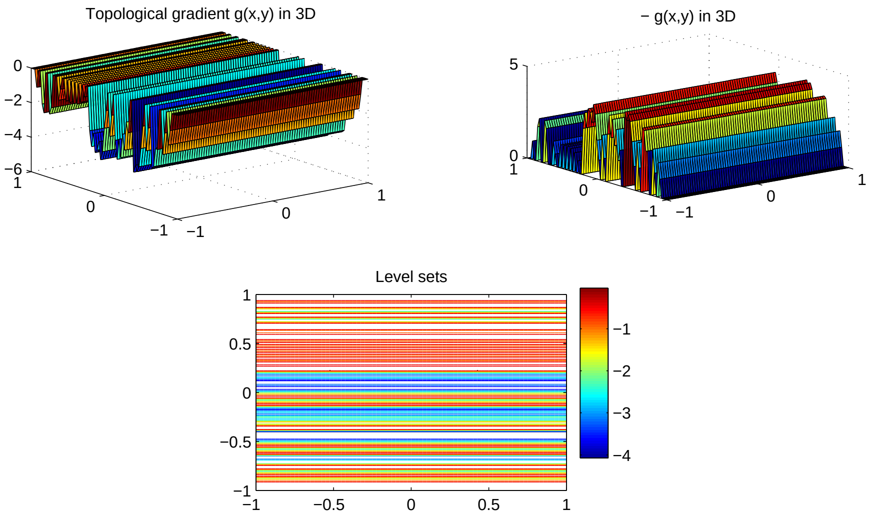
The first example concerns the figures 1 and 2. In this example we choose $r = |x - y|$ and $W_1 = |tan(x^2 + y^2 + 1)|$.





the second example concerns the figures 3 and 4. We choose $r = 1$ and $W_1 = 1/3|x - 2y|$.





Mass Transport translation

Joint works with L. NDIAYE(UCAD in Sénégal)
Is it possible to give a sense this problem as a Mass Transport Problem?
A lot of references: G. Monge 1781, Sudakov, Y. Brenier, L.C. Evans, W. Gangbo; C. Villani, G. Buttazzo, L. Ambrosio, L. Caffarelli, MCCann, E. Carlen, B. Dacoragna, F. Santabroggio; A. Figalli, etc....
$\Omega \subset \mathbb{R}^d$. $\epsilon(t,x)$ porosity of the media , $\rho[kg/m^3]$, V are respectivly the density of the fluid and its velocity,and W is the concentration of polluant.
Basic equations:

- Conservation law

$$\frac{\partial}{\partial t}(\epsilon \rho_t) + div(\rho_t \epsilon V) = 0 \text{ in } \Omega; \qquad (65)$$

- Chemical body conservation law:

$$\frac{\partial}{\partial t}(\epsilon \rho_t W) + div(\rho_t \epsilon V W + \mathcal{V}) = 0 \text{ in } \Omega; \qquad (66)$$

$\mathcal{V}$ dispersion diffusion flow vector
Fick's law

$$\mathcal{V} = -\rho_t \theta(x) \nabla W.$$

Darcy law in porous media (undeformable, weak flow etc.)

$$\epsilon V = q = -\eta(x)(\nabla p + \rho_s g e_z),$$

$\eta(x)$ Hydraulic conductuvity, $p[N/m^2]$ dynamic pressure of the fluid, $g(m/s^2)$ gravitational accelaration e_z unit vector following z.

Case: porosity is a constant

(65) and (66) become

$$\frac{\partial}{\partial t}(\rho_t) + div(\rho_t V) = 0 \text{ in } \Omega; \quad (67)$$

$$\frac{\partial}{\partial t}(\rho_t W) + div(\rho_t V W + \frac{\mathcal{V}}{\epsilon}) = 0 \text{ in } \Omega; \quad (68)$$

Replacing $\mathcal{V}$ by its expression valeur and doing the derivatives of the equation (68), we have:

$$W\frac{\partial}{\partial t}\rho_t + \rho_t\frac{\partial}{\partial t}W + W div(\rho_t V) +$$

$$\rho_t V \cdot \nabla W - div(\rho_t \frac{\theta(x)}{\epsilon}\nabla W) = 0$$

Finally

$$\rho_t\frac{\partial}{\partial t}W + \rho_t V \cdot \nabla W = div(\rho_t \frac{\theta(x)}{\epsilon}\nabla W) \quad (69)$$

Hypothesis: ρ_t of the fluid is a constant

The problem is to solve the following transport equation:

$$\left\{ \begin{array}{ll} \frac{\partial}{\partial t}W + \xi \cdot \nabla W = div(\frac{\theta(x)}{\epsilon}\nabla W) \quad sur \quad]0,T[\times\Omega \\ W(0,x) = W_0(x) \end{array} \right. \tag{70}$$

See for example, Calculus of Variations and non linear PDE 2005 eds: B. Dacoragna and P. Marcellini (L. Ambrosio , L. Caffarelli, M. Crandall, L.C. Evans and N. Fusco) etc.

If $div(\frac{\theta(x)}{\epsilon}\nabla W) \in L^1_{loc}([0,T] \times \mathbb{R}^d$,

$$W_t(X(t,x)) = W_0 + \int_0^t div(\frac{\theta}{\epsilon}\nabla W(X(s,x)) \ \forall t \in [0,T].$$

Particular case

$\theta \equiv cte$ or $\nabla W \perp \nabla(\frac{\theta(x)}{\epsilon})$

$$\frac{\partial}{\partial t}W + \xi \cdot \nabla W - \frac{\theta(x)}{\epsilon}\triangle W = 0$$

Remark!

For $rot\xi = 0$,

$$\xi(t,x) = \nabla\Big(\varphi(t,x)\Big),$$

$$\frac{\partial}{\partial t}W + \nabla\varphi \cdot \nabla W - \frac{\theta(x)}{\epsilon}\triangle W = 0.$$

Let us consider the measure
$\nu(dx) = \exp(-\frac{\epsilon}{\theta(x)}\varphi(t,x))vol(dx)$, with $W(0,x) = W_0(x)$ at $t = 0$ et $W(,x) = W_1(x)$ at $t = 1$ are given.

We define $\mu_0 = \frac{\epsilon}{\theta(x)}W_0\nu$, $\mu_1 = \frac{\epsilon}{\theta(x)}W_1\nu$
and the vector

$$\xi(t,x) = \frac{\nabla W}{(1-t)\mu_0 + t\mu_1},$$

is associated to the family of measures

$$\mu_t = (1-t)\mu_1 + t\mu_0 = \frac{\epsilon}{\theta(x)}\nu W_t$$

One verifies

$$\partial_t\mu = \mu_0 - \mu_1 = \frac{\epsilon}{\theta(x)}(W_0 - W_1)\nu$$

$$\begin{aligned}
\nabla\cdot(\mu_t\xi(t,\cdot)) &= \nabla\cdot\Big(\nabla W\exp(-\frac{\epsilon}{\theta(x)}\varphi(t,x))vol\Big) \\
&= \exp(-\frac{\epsilon}{\theta}\varphi)(\triangle W - \frac{\epsilon}{\theta}\nabla\varphi\cdot\nabla)vol \\
&= \frac{\epsilon}{\theta(x)}(\frac{\theta(x)}{\epsilon}\triangle W - \nabla\varphi\cdot\nabla)\nu \\
&= \frac{\epsilon}{\theta(x)}(W_1 - W_0)\nu = -\partial_t\mu
\end{aligned}$$

Finally we have

$$\partial_t \mu + \nabla \cdot (\mu_t \xi(t,\cdot)) = 0$$

Minimize a coast of transport under the following constraints:

$$\begin{cases} \partial_t \mu(t,x) + \nabla \cdot (\mu(t,x)\xi(t,x)) &= 0 \quad x \in \mathbb{R}^d,\ t > 0; \\ \xi(t,x) &= \nabla\Big(\varphi(t,x)\Big) \\ \mu(0,x) &= \mu_0(x) \quad \mu_0 \in L^1(\mathbb{R}^d),\ \mu_0 \geq 0. \end{cases} \tag{71}$$

When φ is the first variation of the Euler-Lagrange of an integral functional ϕ i.e.

$$\varphi = \frac{\delta \phi}{\delta W} \quad \text{with} \quad \phi(u) = \int \mathcal{L}(x, u, \nabla u)\, dx,$$

And then it suffices to solve the following problem:

$$\begin{cases} \partial_t \mu(t,x) + \nabla \cdot (\mu(t,x)\xi(t,x)) &= 0 \quad x \in \mathbb{R}^d,\ t > 0; \\ \xi(t,x) &= \nabla\Big(\varphi(t,x)\Big) \\ \mu(0,x) &= \mu_0(x) \quad \mu_0 \in L^1(\mathbb{R}^d),\ \mu_0 \geq 0. \end{cases} \tag{72}$$

Numerical simulations in Process

A question: What's happen when the porosity ϵ depends on x and t?

Invitation to the CIMPA international school in Dakar (UCAD)

5 april to 15 april 2011

see website for additional information

www.ucad.sn/cimpa

or

http://cimpadakar2011.ucad.sn

END

THANK YOU