ON THE PULLBACK EQUATION

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1) Bandyopadhyay S. and Dacorogna B., On the pullback equation $\varphi^* (g) = f$, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009), 1717-1741.

2) Bandyopadhyay S., Dacorogna B. and Kneuss O., The pullback equation for degenerate forms, *Disc. Cont. Dyn. Syst. Series A*, **27** (2010), 657-691.

3) Dacorogna B. and Kneuss O., Divisibility in Grassmann algebra, to appear in Linear and Multilinear Algebra.

I) Introduction

We discuss the existence of a diffeomorphism

$$\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

such that

$$\varphi^* (g) = f$$

where, $1 \leq k \leq n$,

$$f, g : \mathbb{R}^n \longrightarrow \Lambda^k (\mathbb{R}^n) \approx \mathbb{R}^{\binom{n}{k}}$$

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$f, g \neq 0$, are closed differential forms (i.e. $df = dg = 0$), $2 \leq k \leq n$

$$g = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} (x) \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

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$$g = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} (x) \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and similarly for f. The meaning of (1) is that

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} (\varphi (x)) \, d\varphi^{i_1} \wedge \cdots \wedge d\varphi^{i_k}$$

$$= \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} (x) \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} .$$

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we get that (2) is equivalent to

$$\sum_{p=1}^{n} g_p(\varphi(x)) \frac{\partial \varphi^p}{\partial x^i} = f_i \quad i = 1, \cdots, n$$

which is a linear (in the derivatives) first order system of $\binom{n}{1} = n$ pdes.

$k = 2$ $f, g : \mathbb{R}^n \to \Lambda^2(\mathbb{R}^n) \approx \mathbb{R}^{\binom{n}{2}} = \mathbb{R}^{\frac{n(n-1)}{2}}$

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The equation $\varphi^*(g) = f$ becomes

$$\sum_{1\le p<q\le n} g_{pq}(\varphi(x))\, d\varphi^p \wedge d\varphi^q = \sum_{1\le i<j\le n} f_{ij}(x)\, dx^i \wedge dx^j.$$

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we get that (3) is equivalent, for every $1 \le i < j \le n$, to

$$\sum_{1 \le p < q \le n} g_{pq}(\varphi(x)) \left(\frac{\partial \varphi^p}{\partial x^i} \frac{\partial \varphi^q}{\partial x^j} - \frac{\partial \varphi^p}{\partial x^j} \frac{\partial \varphi^q}{\partial x^i} \right) = f_{ij}$$

which is a non-linear homogeneous of degree 2 (in the derivatives) first order system of $\binom{n}{2}$ pdes.

k $f, g : \mathbb{R}^n \to \Lambda^k (\mathbb{R}^n) \approx \mathbb{R}^{\binom{n}{k}}$.

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Questions

1) Local existence

2) Global existence

3) Regularity

4) Dirichlet (or Cauchy) data

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Plan of the talk

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II) Historical background ($k = 2$ and $k = n$)

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VIII) Ideas of the proof

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Theorem (Darboux, 1882) Let $n = 2m$ and $x_0 \in \mathbb{R}^n$. Let ω_m be the *standard symplectic form*

$$g = \omega_m = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i}$$

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Then there exist a neighbourhood V of x_0 and $\varphi \in \operatorname{Diff}(V; \mathbb{R}^n)$ such that

$$\varphi^*(\omega_m) = f \text{ in } V \quad \text{and} \quad \varphi(x_0) = x_0 .$$

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(i) $f, g \in C^{r,\alpha}\left(\overline{\Omega}\right)$ and

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(ii) There exists $\varphi \in \text{Diff}^{r+1,\alpha}\left(\overline{\Omega}\right)$ satisfying

$$\varphi^*(g) = f \text{ in } \Omega \quad \text{and} \quad \varphi = id \text{ on } \partial\Omega$$

meaning that

$$\begin{cases} g(\varphi(x)) \det \nabla\varphi(x) = f(x) & x \in \Omega \\ \varphi(x) = x & x \in \partial\Omega. \end{cases}$$

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2) Cupini-Dacorogna-Kneuss have studied the degenerate case where f is allowed to change sign (of course, in this case, the map cannot be a diffeomorphism).

III) Darboux theorem with optimal regularity ($k = 2$)

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Theorem (Bandyopadhyay-Dacorogna, 2009) Let $n = 2m$ and $x_0 \in \mathbb{R}^n$. Let ω_m be the *standard symplectic form*

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Let $r \geq 0$ and $0 < \alpha < 1$. Let f be a $2-$form. Then the two following statements are equivalent.

(i) The $2-$form f is closed (i.e. $df = d\omega_m = 0$), $f \in C^{r,\alpha}$ and verifies

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Then there exists $\varphi \in \operatorname{Diff}^{r+1,\alpha}\left(\overline{\Omega}\right)$ satisfying

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(iii) The condition

$$\text{rank}\,[tg + (1-t)\,f] = n \quad \text{in } \Omega \text{ and for every } t \in [0,1]$$

can be weakened, replacing the linear homotopy by a non-linear one.

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where $\mathcal{D}_2(\Omega)$ is the set of 2−harmonic field namely

$$\mathcal{D}_2(\Omega) = \left\{ \begin{array}{c} \psi \in C^1 : d\psi = 0, \ \delta\psi = 0 \\ \text{and } \nu \wedge \psi = 0 \text{ on } \partial\Omega \end{array} \right\}.$$

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The dimension of this space is the Betti number

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V) Darboux type theorem in the degenerate cases

Theorem (Bandyopadhyay-Dacorogna-Kneuss, 2010)
Let $2 \leq 2l < n$ and $x_0 \in \mathbb{R}^n$. Let ω_l be the *standard symplectic form of* $\text{rank}\,\omega_l = 2l$

$$g = \omega_l = \sum_{i=1}^{l} dx^{2i-1} \wedge dx^{2i}$$

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Then there exist a neighbourhood V of x_0 and $\varphi \in \operatorname{Diff}^{r,\alpha}(V; \mathbb{R}^n)$ such that

$$\varphi^* (\omega_l) = f \quad \text{in } V \quad \text{and} \quad \varphi(x_0) = x_0 .$$

VI) The case of $(n-1)$ −forms

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or equivalently

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$$f = *\left(\nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1}\right) \quad \text{and} \quad \varphi(x_0) = x_0\,.$$

VII) The case of $k-$forms when $3 \leq k \leq n-2$

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Note that f is a $k = (2l+m)$ $-$form.

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and recover u_t through the overdetermined algebraic relation

$$u_t \lrcorner f_t = \omega.$$

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This requires very fine properties of Hölder continuous functions.