Principal eigenvalue in an unbounded domain with a potential

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Introduction

In this talk, we are mainly concerned with the existence of a positive principal eigenvalue for the following problem

$$-\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u \quad \text{in} \ \ \Omega, \tag{1}$$

where

1. Ω is an unbounded smooth domain in $\mathbb{R}^N$,
2. $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$, $1 < p < +\infty$, is the p-Laplacian,
3. λ is the eigenvalue parameter,
4. V and m are weight functions possibly indefinite whose properties will be specified later.

Introduction

By principal eigenvalue, we mean $\lambda > 0$ such that (1) admits a nontrivial u, with $u \geq 0$, in a suitable weak sense.

Case $V \equiv 0$

Many works have been devoted to this case due to the importance of the validity of the weighted Poincaré inequality. It was prove the existence of principal eogenvalue

$$\lambda_1(m) := \inf\{E(u) = \int_\Omega |\nabla u|^p, u \in W \text{ and } \int_\Omega m|u|^p = 1\}$$

where W is a suitable Sobolev space.

Case $V \geq 0$

This case is technically very similar to the case $V \equiv 0$, as the energy fonction associated

$$E_V(u) = \int_\Omega (|\nabla u|^p + V|u|^p)$$

has similar properties withe energy E defined above.

Case when V changes sign

Problem (1) with V changing sign were recently considered

1. Case $m \equiv 1$, see [19]
2. m indefinite, $p = 2$ and additional hypothesis on V and m, see [10, 14]
3. m indefinite, see [5, 6, 9, 11, 17]

Arguments are based on variational method that will be also used here.

Preliminary

Throughout this communication, Ω is an unbounded smooth domain of $\mathbb{R}^N$; $\Omega_R = \Omega \cap B(o,R)$ is smooth and $\Omega'_R = \Omega \backslash B(o,R)$.

We assume the validity on Ω of the weighted Poincaré inequality of the form :

$$(H_g) \qquad \int_\Omega |u|^p \leq K(\Omega, g) \int_\Omega |\nabla u|^p \quad \forall u \in C_c^\infty(\Omega),$$

for some function $g \geq 0, \not\equiv 0$ suitably related to the weight m and for some positive constant $K(\Omega, g)$

Remark on hypothese (H_g)

1. When $N \leq p$, an inequality of the form (H_g) with $g \not\equiv 0$ never hold for $\Omega = \mathbb{R}^N$.
2. We consider $N \leq p$ and set $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : -\ln(2 + |x_1|) < x_2 < \ln(2 + |x_1|)\}$. Take $m = m_1(x_1, x_2) = \dfrac{1}{(1 + |x_1|)^\epsilon}$ for $\epsilon > 0$. We claim that (H_g) holds for $g(x_1, x_2) = \dfrac{1}{[\ln(2 + |x_1|)]^p}$ with $K(\Omega, g) = 2^p$.
3. Take $N > p$, $m = m_1 \in L^{N/p}(\Omega)$, $\Omega = \mathbb{R}^N$ and $g(x) = \dfrac{1}{1 + |x|^p}$. Then (H_g) holds by using Hardy inequality.

Admissible function

We will say that a nonnegative function ρ is "admissible" if it satisfies the following local integrablility condition :

$$(H_0) \qquad \rho \in L^s(\Omega_R) \ \ \forall R > 0,$$

for some $s > \dfrac{N}{p}$, when $N \geq p$ and $s = 1$ when $N < p$.

Sobolev weighted space

Associated with such function ρ, we define, when $\rho \not\equiv 0$, the weighted space W_ρ as the closure of $C_c^\infty(\Omega)$ with respect to the norm

$$||u||_{W_\rho} := \left[\int_\Omega (|\nabla u|^p + \rho|u|^p) \right]^{1/p}.$$

The following imbeddings hold (cf. e.g [7, 11]) :

1. $W_\rho \hookrightarrow D^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ if $N > p$
2. $W_\rho \hookrightarrow W^{1,p}(\Omega_R)$ for all p.

Decomposition of the weight function

We write the weight m in the form $m = m_1 - m_2$, with m_1, m_2 admissible functions. Note that the decomposition $m = m_1 - m_2$ does not necessarily coincide with the usual decomposition $m = m^+ - m^-$, where $m^\pm = \max(\pm m, 0)$.

Principal eigenvalue in low dimensions

In this section, we will assume that $N \leq p$ and consider successively a weight m either nonnegative or indefinite in Ω with the following hypothesis

(H_1) : There exists an admissible function g such that (H_g) holds and

$$m_1(x) \leq \theta(x)g(x) \quad \text{a.e in} \quad \Omega,$$

where the function θ verifies

$$||\theta||_{L^\infty(\Omega'_R)} \to 0 \text{ as } R \to \infty.$$

(H_2) : There exists a positive constant c with, $0 \leq c < \frac{1}{K(\Omega,g)}$ such that

$$|V(x)| \leq cg(x) \quad \text{a.e in} \quad \Omega,$$

Compact imbedding

Lemma

Under the hypothese (H_1)one has the following compact imbedding $W_g \hookrightarrow\hookrightarrow L^p(m_1, \Omega)$, where $L^p(m_1, \Omega)$ denotes the L^p space on Ω with weight m_1.

Case m is nonnegative, i.e $m = m_1$

By a solution u of (1), we mean a weak solution, i.e. $u \in W_g$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v + \int_\Omega V(x)|u|^{p-2}uv = \lambda \int_\Omega m(x)|u|^{p-2}uv, \quad \forall\ v \in W_g. \tag{2}$$

Note that every term in (2) is well defined.

The following theorem concerns the case m is a nonnegative function.

Case m is nonnegative, i.e $m = m_1$

Theorem

Let $N \leq p$ and $m = m_1$, with m_1 admissible and $m \not\equiv 0$. Assume that m_1 satisfies (H_1) and V satisfies (H_2). Then (1) admits a principal eigenvalue $\lambda_1(V, m)$ with corresponding eigenfunction belongs to W_g.

Ideas of the proof

1. Consider the following C^1 functionals

$$E_V(u) = \int_\Omega (|\nabla u|^p + V|u|^p), \qquad B(u) = \int_\Omega m_1|u|^p.$$

2. E_V is coercive on W_g.
3. Set

$$\lambda_1(V, m_1) := \inf\{E_V(u), u \in W_g \text{ and } B(u) = 1\}.$$

- Prove that this infinimum is achieved at some nonegative function.
- Using Lagrange multipliers rule yields to the conclusion that $\lambda_1(V, m_1)$ is a principal eigenvalue.

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The weight m is in the form $m = m_1 - m_2$

Next we consider an arbitrary weight m in the form $m = m_1 - m_2$, where m_1, m_2 are nonnegative functions. For fixed λ, We consider the following eigenvalue problem with parameter $\beta(\lambda)$

$$-\Delta_p u + (V + \lambda m_2)|u|^{p-2}u = \beta(\lambda) m_1 |u|^{p-2}u. \tag{3}$$

It is clear that λ is an eigenvalue of (1) if and only if λ is a fixed point of $\beta(\lambda)$ where $\beta(\lambda)$ is an eigenvalue of (3). Searching such a λ is our goal in the next paragraph.

Let start to prove the following result.

The weight m is in the form $m = m_1 - m_2$

Theorem

Suppose $N \leq p$ and let $m = m_1 - m_2$ with m_1, m_2 admissibles. Assume that m_1 satisfies (H_1) and V satisfies (H_2). Assume also that

$$(H_3) \qquad m^+ \not\equiv 0$$

Then (1) admits a principal eigenvalue $\lambda_1(V, m_1 - m_2)$, having an eigenfunction which belongs to W_{g+m_2} (where g is provided by (H_1)).

Ideas of the proof

First step : Existence of principal eigenvalue

We will first prove that problem (3) admits a principal eigenvalue $\lambda_1(V+\lambda m_2, m_1)$ by considering the following functionals :

$$A_\lambda(u) := \int_\Omega [|\nabla u|^p + (V(x)+\lambda m_2)|u|^p] \qquad \text{and set}$$

$$\beta(\lambda) := \inf\{A_\lambda(u) : \quad u \in W_{g+m_2} \quad \text{and} \quad B(u) = 1\}.$$

1. A_λ is coercive on W_{m_2+g}.
2. Lagrange multiplier rule yields to the conclusion that $\beta(\lambda)$ is a principal eigenvalue.

Ideas of the proof

Second step : Existence of a fixed point for $\beta(\lambda)$

Since for $\lambda > 0$ and for all $u \in W_{g+m_2}$ with $B(u) = 1$,

$$A_\lambda(u) = E_V(u) + \lambda \int_\Omega m_2(x)|u|^p \geq E_V(u) \geq \lambda_1(V, m_1),$$

where $\lambda_1(V, m_1)$ is defined in Theorem 2, one deduces that

$$\beta(\lambda) \geq \lambda_1(V, m_1).$$

Consequently, $\beta(\lambda) \geq \lambda$ for $0 < \lambda \leq \lambda_1(V, m_1)$.

Ideas of the proof

Second step : Existence of a fixed point for $\beta(\lambda)$

(H_3) implies that there exists a nonempty open subset, with positive measure, $\Omega_0 \subset \Omega$ such that

$$m = m_1 - m_2 > 0 \quad \text{a.e in} \quad \Omega_0. \tag{4}$$

Let $u_0 \in W_{g+m_2}$ with $u_0 \not\equiv 0$ and *supp* $u_0 \subset \Omega_0$. Then for λ sufficiently large

$$A_\lambda(u_0) - \lambda \int_\Omega m_1 |u_0|^p = \int_\Omega |\nabla u_0|^p - \lambda \int_\Omega m|u_0|^p < 0. \tag{5}$$

From (5) (taking $v_0 = u_0/[B(u_0)]^{1/p}$ if necessary), one deduces that $\beta(\lambda) < \lambda$ for λ sufficiently large.

Ideas of the proof

Second step : Existence of a fixed point for $\beta(\lambda)$

Moreover β is a concave u.s.c function from $\mathbb{R}_0^+$ to $\mathbb{R}$ (since it is the infimum of a family of affine functions of λ), and so in particular β is continuous on $\mathbb{R}_0^+$. It follows that β admits a positive fixed point λ_0 (ie $\beta(\lambda_0) = \lambda_0$), and the theorem is proved.

The weight m is nonegative

As in the previous section, we will investigate the case when the weight m is a positive function and when it is indefinite. We assume that $N > p$. Let us begin with the first result when the weight is nonegative.

Theorem

Suppose $N > p$ and let $m = m_1$ with m_1 admissible and $m_1 \not\equiv 0$. Assume that

(H_4) $$m_1 \in L^{N/p}(\Omega)$$

(H_5) $$V \in L^{N/p}(\Omega) \quad \text{and} \quad ||V||_{L^{N/p}(\Omega)} \leq \frac{1}{c_1^p},$$

where c_1 is the constant of the imbedding of $D^{1,p} \hookrightarrow L^{P^*}(\Omega)$. Then the problem (1) admits a principal eigenvalue $\lambda_1(V, m_1)$, having an eigenfunction belonging to $D^{1,P}(\Omega)$.

Ideas of the proof

The proof of Theorem 4 relies on the following lemma

Lemma

[4] Assume that $N > p$. The functional

$$u \mapsto \int_\Omega \omega |u|^q$$

is well defined and weakly continuous in $D^{1,p}(\Omega)$, for $1 \leq q < p^*$ and $\omega \in L^{(p^*/q)'}(\Omega)$.

Next, we turn to the case $m = m_1 - m_2$ with nontrivial positive part m^+ (i.e $m^+ \not\equiv 0$ in Ω).

Theorem

Suppose $N > p$ and let $m = m_1 - m_2$ with m_1, m_2 admissible functions. Assume that m satisfies (H_3) and m_1 satisfies (H_4). Assume further that V satisfies (H_5). Then the problem (1) admits a principal eigenvalue $\lambda_1(V, m_1 - m_2)$, having an eigenfunction belonging to W_{m_2}.

Some properties of the eigencurve $\beta(\lambda)$

We give below some properties of the function $\beta(\lambda)$.

Theorem

Let $\lambda_n \to \lambda$ and $(\beta(\lambda_n), \varphi_n)$, $(\beta(\lambda), \varphi_\lambda)$ be the corresponding eigenpairs for problem (3). Then $\beta(\lambda_n) \to \beta(\lambda)$ and $\varphi_n \to \varphi_\lambda$ in W_{m_2}. Moreover $\beta(\lambda)$ is differentiable and for any $\lambda_0 \in \mathbb{R}$, we have

$$\beta'(\lambda_0) = \int_\Omega m_2 \varphi_{\lambda_0}^p \tag{6}$$

Remark

In Section 3 (respectively Section 4), hypothese (H_2) in Theorem 3.1 and Theorem 3.3 (resp. (H_5) in Theorem 4.1 and Theorem 4.3) guarantees the positivity of the principal eigenvalue. In what follows, we aim to weak these hypotheses, that will allow the principal eigenvalue to be either positive or negative.

Remark

For that let us introduce the following number (cf. [9, 17])

$$\alpha(V,m) := \inf\{E_V(u), u \in D^{1,p}(\Omega), \int_\Omega m|u|^p = 0 \text{ and } \int_\Omega |u|^p = 1\}.$$

It is easily seen that $\alpha(V,m) = \infty$ if and only if $m > 0$ in Ω. Futhermore $\alpha(V,m)$ is achieved whenever it is finite

When $m \geq 0$, $\alpha(V, m)$ is somehow close to the first eigenvalue of

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u & = \lambda|u|^{p-2}u \text{ in } \Omega_0 \\ u & = 0 \text{ in } \partial\Omega_0 \end{cases}$$

where $\Omega_0 = \Omega \backslash supp(m)$

Theorem

Suppose $N > p$ and let $m = m_1 - m_2$ with m_1, m_2 admissible functions. Assume that m satisfies (H_3) and (H_4). Assume further that V changes sign and satisfies

$$(H_5') \qquad V \in L^{N/p}(\Omega).$$

Then the problem (1) admits a principal eigenvalue $\lambda_1(V, m_1 - m_2)$, having an eigenfunction $\varphi \in D^{1,p}(\Omega)$ if $\alpha(V, m) > 0$.

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