Solving a linear conservation law subject to initial and final conditions: An inverse problem.

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Introduction, Motivation

Modern medical imaging provides a great amount of information. Allow to study the anatomy and physiological functions in both space and time.

Example: Cardiac Magnetic Resonance Imaging (MRI), cross-section images used to map the heart in 3D at a collection of discrete time samples over the cardiac cycle.

Challenge: From partial observations reconstruct the cardiac cycle.

Image tracking

Given a pair of images, the aim of image tracking is: find a sequence of intermediate images, such that the first image in the sequence is equal to the first given image (starting image) and the last image is equal to the second given image (ending image). For this, find a precise process to interpolate the image intensity functions between the two images.

Let ρ be the intensity image function, and u be the velocity of the apparent motion of brightness pattern. An image is discribed for example by its gray-value map. Motion of points of the image under the velocity field u, the gray values $\rho(t, X(t,x))$ are constant along the trajectories $X(t,x)$. This is expressed with the transport equation (*optical flow equation*):

$$\partial_t \rho(t, X(t,x)) + (\, u \mid \nabla_X \rho(t, X(t,x)) \,) = 0.$$

Weaker assumption: replaces the intensity preserving by a mass preserving conservation law (*extend optical flow equation*).

$$\partial_t \rho + (\, u \mid \nabla_x \rho \,) + \mathrm{div}\,(u)\rho = 0.$$

One question of image interpolation is: Given an (approximate) velocity field, solve the conservation law subject to initial and final conditions.

A priori ill posed problem

Characterize the range of the PDE operator subject to constraint, which is a quite difficult question
Idea: solve the problem in the least-squares sense.

For the conservation law operator, if the velocity field $\boldsymbol{u}$ is sufficiently regular, then a space-time $\boldsymbol{L^2}$ least-squares formulation provides the renormalized solution. Taking advantage of that result, it is easy to express the conservation law operator subject to initial and final conditions problem in the $\boldsymbol{L^2}$ least-squares framework.

Plan

1. Least-squares and functional spaces

2. Existence and uniqueness results

3. Numerical experiments

1. Least-squares and functional spaces

$\Omega \subset \mathbb{R}^d$ bounded domain, boundary $\partial\Omega$ sufficiently regular. For $T > 0$ given, set $Q = \Omega \times]0, T[$. Advection velocity $u : Q \longrightarrow \mathbb{R}^d$ and $f : Q \longrightarrow \mathbb{R}$ a given source term. The velocity u has the following regularity

$$u \in L^\infty(Q)^d \text{ and } \operatorname{div} u \in L^\infty(Q).$$

Let

$$\Gamma_- = \{x \in \partial\Omega : (\, u(x,t) \,|\, n(x) \,) < 0\}$$

where $n(x)$ is the outer normal to $\partial\Omega$ at point x. For simplicity Γ_- do not depend on t.

Problem: find $c : Q \longrightarrow \mathbb{R}$ such that

$$\partial_t c + \operatorname{div}(c\, u) = f \quad \text{in } Q, \tag{1}$$

with initial, final, and inflow boundary conditions

$$c(x,0) = \quad c_0(x) \qquad \text{for } x \text{ in } \Omega \tag{2}$$

$$c(x,T) = \quad c_T(x) \qquad \text{for } x \text{ in } \Omega \tag{3}$$

$$c(x,t) = \; c_1(x,t) \qquad \text{for } x \text{ on } \Gamma_- \times (0,T). \tag{4}$$

In the sequel assume (to simplify) that $c_1 = 0$ or $\Gamma_- = \emptyset$.

Hilbert spaces

For $u \in L^{\infty}(Q)^d$, with $\mathrm{div}\ u \in L^{\infty}(Q)$, let

$$\widetilde{u} = (1, u_1, u_2, \ldots, u_d)^t \in L^{\infty}(Q)^{d+1},$$

and for φ defined on Q, regular enough, set

$$\widetilde{\nabla}\varphi = \left(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \ldots, \frac{\partial\varphi}{\partial x_d}\right)^t,$$

and

$$\widetilde{\mathrm{div}}(\widetilde{u}\,\varphi) = \frac{\partial\varphi}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(u_i\,\varphi).$$

Finally $\widetilde{n}$ denotes the outward unit vector on ∂Q.

Theorem 1. (CHEN-FRID) *If $u \in L^{\infty}(Q)^d$, and $\mathrm{div}\ u \in L^{\infty}(Q)$, then the normal trace of u, $(\,\widetilde{u} \,|\, \widetilde{n}\,)$ is in $L^{\infty}(\partial Q)$.*

Let now

$$\begin{aligned}\partial Q_- &= \{(x,t) \in \partial Q, (\,\widetilde{u}\,|\,\widetilde{n}\,) < 0\} \\ &= \Gamma_- \times (0,T) \ \cup \ \Omega \times \{0\}, \\ \partial Q_+ &= \{(x,t) \in \partial Q, (\,\widetilde{u}\,|\,\widetilde{n}\,) > 0\} \\ &= \Gamma_+ \times (0,T) \ \cup \ \Omega \times \{T\},\end{aligned}$$

and set

$$c_b(x,t) = \begin{cases} c_0(x) & \text{if} \quad (x,t) \in \Omega \times \{0\} \\ 0 & \text{if} \quad (x,t) \in \Gamma_- \times (0,T). \end{cases} \tag{5}$$

Assume that

$$c_b \in L^2(\partial Q_-), \quad c_T \in L^2(\Omega),$$

For $\varphi \in \mathcal{D}_0(\overline{Q}) = \{\varphi \in \mathcal{D}(\overline{Q}),\ \varphi|_{\Gamma_+ \times (0,T)} = 0\}$, consider the norm

$$\|\varphi\|_{H(u,Q)} = \left(\|\varphi\|^2_{L^2(Q)} + \|\widetilde{\mathrm{div}}(\widetilde{u}\,\varphi)\|^2_{L^2(Q)} + \int_{\partial Q_-} |\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,|\varphi^2\, d\widetilde{\sigma} \right)^{1/2}, \tag{6}$$

Define the space $H(u,Q)$ as the closure of $\mathcal{D}_0(\overline{Q})$ for this norm:

$$H(u,Q) = \overline{\mathcal{D}_0(\overline{Q})}^{H(u,Q)}$$

If u is regular enough, then

$$\begin{aligned} H(u,Q) &= \{\rho \in L^2(Q), \widetilde{\operatorname{div}}(\widetilde{u}\,\rho) \in L^2(Q), \\ &\quad \rho|_{\partial Q_-} \in L^2(\partial Q_-, |\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,|\,d\widetilde{\sigma})\rho|_{\Gamma_+\times(0,T)} = 0\} \end{aligned}$$

This is a Meyers-Serrin theorem.

Trace result for $H(u, Q)$

$$\begin{array}{rcl} \gamma : \; H(div, Q) & \longrightarrow & H^{-\frac{1}{2}}(\partial Q) \\ v & \mapsto & \gamma v = (\,\tilde{n}\,|\,v\,)\,|_{\partial Q}, \end{array}$$

Associated Green formula:

$$\int_Q \widetilde{\mathrm{div}}(v)\psi + \left(\, v \,|\, \tilde{\nabla}\psi \,\right) \, dx\, dt = <\, (\, v \,|\, \tilde{n}\,)\, , \psi \, >_{H^{-\frac{1}{2}}(\partial Q); H^{\frac{1}{2}}(\partial Q)},$$

$\forall \psi \in H^1(Q)$. Choose $v = \tilde{u}\rho$ gives:

$$\int_Q \widetilde{\mathrm{div}}(\tilde{u}\rho)\psi + \left(\, \tilde{u} \,|\, \tilde{\nabla}\psi \,\right) \rho \, dx\, dt = <\, \rho\, (\, \tilde{u} \,|\, \tilde{n}\,)\, , \psi \, >_{H^{-\frac{1}{2}}(\partial Q); H^{\frac{1}{2}}(\partial Q)},$$

$\forall \psi \in H^1(Q)$.

We are interested in a trace operator at time T. Consider the bilinear form

$$L : \mathcal{D}_0(\overline{Q}) \times \mathcal{D}_0(\overline{Q}) \subset H(u,Q) \times H(u,Q) \longrightarrow \mathbb{R}$$

defined for all $\varphi, \psi \in \mathcal{D}(\overline{Q})$ by:

$$\begin{aligned} L(\varphi,\psi) \;&=\; \int_Q \widetilde{\mathrm{div}}(\tilde{u}\varphi)\psi + \left(\, \tilde{u} \,|\, \tilde{\nabla}\psi \,\right) \varphi \, dx\, dt + \\ &\quad \int_{\partial Q_-} |\, (\, \tilde{u} \,|\, \tilde{n}\,)\,| \varphi\psi \, d\tilde{\sigma}. \end{aligned} \tag{7}$$

Choose $\psi = \varphi$ and integrate by parts the first term of the first integral on the right hand side of (7), we obtain

$$L(\varphi,\varphi) = \|\varphi(\cdot,T)\|^2_{L^2(\Omega)}.$$

Since $\boldsymbol{u}$ and $\widetilde{\boldsymbol{u}}$ have the same regularity, Chen-Frid theorem allows to define the measure $|\,(\,\widetilde{\boldsymbol{u}}\,|\,\widetilde{\boldsymbol{n}}\,)\,|d\widetilde{\boldsymbol{\sigma}}$ on $\partial \boldsymbol{Q}_-$ and we have the following estimate for the bilinear form $\boldsymbol{L}$:

$$\begin{aligned} |\boldsymbol{L}(\varphi,\psi)| \;&\leq\; \|\widetilde{\mathbf{div}}(\widetilde{\boldsymbol{u}}\varphi)\|_{L^2(Q)}\|\psi\|_{L^2(Q)} \\ &+\|\widetilde{\mathbf{div}}(\widetilde{\boldsymbol{u}}\psi) - \widetilde{\mathbf{div}}(\widetilde{\boldsymbol{u}})\psi\|_{L^2(Q)}\|\varphi\|_{L^2(Q)} \\ &+\|\varphi\|_{L^2(\partial Q_-,|(\,\widetilde{u}\,|\,\widetilde{n}\,)|d\widetilde{\sigma})}\|\psi\|_{L^2(\partial Q_-,|(\,\widetilde{u}\,|\,\widetilde{n}\,)|d\widetilde{\sigma})}, \end{aligned} \tag{8}$$

therefore

$$|\boldsymbol{L}(\varphi,\psi)| \leq (1 \,+\, \|\mathrm{div}(\boldsymbol{u})\|_{L^\infty(Q)})\|\varphi\|_{H(u,Q)}\|\psi\|_{H(u,Q)}.$$

Extend L by continuity to $H(u,Q) \times H(u,Q)$ then:

Proposition 2. *Under the assumption* $u \in L^\infty(Q)^d$, *and* $\mathrm{div}\ u \in L^\infty(Q)$ *there exists a linear continuous trace operator*

$$\begin{array}{rccc} \gamma_{\widetilde{n}}: & H(u,Q) & \longrightarrow & L^2(\partial Q, |\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,|d\widetilde{\sigma}) \\ & \varphi & \longmapsto & \gamma_{\widetilde{n}}\varphi = \varphi_{|\partial Q}, \end{array}$$

which can be localized as:

$$\begin{array}{rccc} \gamma_{\widetilde{n}_+}: & H(u,Q) & \longrightarrow & L^2(\Omega) \\ & \varphi & \longmapsto & \gamma_{\widetilde{n}_+}\varphi = \varphi(\cdot,T). \end{array}$$

Finally define the space

$$H_0 = H_0(u,Q) = \overline{\mathcal{D}(Q)}^{H(u,Q)}.$$

Curved Poincaré inequality

Theorem 3. CURVED POINCARÉ INEQUALITY *If* $\boldsymbol{u} \in \boldsymbol{L}^{\infty}(\boldsymbol{Q})^{\boldsymbol{d}}$ *and* $\mathbf{div}\ \boldsymbol{u} \in \boldsymbol{L}^{\infty}(\boldsymbol{Q})$*, the semi-norm on* $\boldsymbol{H}(\boldsymbol{u}, \boldsymbol{Q})$ *defined by*

$$|\boldsymbol{\rho}|_{1,\boldsymbol{u}} = \left(\int_{\boldsymbol{Q}} (\widetilde{\mathbf{div}}(\widetilde{\boldsymbol{u}}\boldsymbol{\rho})^2 \boldsymbol{dx}\ \boldsymbol{dt} + \int_{\partial \boldsymbol{Q}_-} |\,(\,\widetilde{\boldsymbol{u}}\,|\,\widetilde{\boldsymbol{n}}\,)\,| \boldsymbol{\rho}^2\ \boldsymbol{d}\widetilde{\boldsymbol{\sigma}} \right)^{1/2} \tag{9}$$

is a norm, equivalent to the norm given on $\boldsymbol{H}(\boldsymbol{u}, \boldsymbol{Q})$*.*

Henceforth the space $\boldsymbol{H}(\boldsymbol{u}, \boldsymbol{Q})$ is equipped with the norm $|\boldsymbol{\varphi}|_{1,\boldsymbol{u}}$.

Remark 4. *a) Using the above result, if* $c_b = 0$*, the semi-norm*

$$|\rho|_{1,u} = \left(\int_Q (\widetilde{\mathrm{div}}(\tilde{u}\rho)^2 dx\, dt\right)^{1/2}$$

in a norm on H_0 *which is equivalent to the usual norm on* $H(u, Q)$*.*
b) As an easy consequence of the above arguments, for any $\rho \in H(u, Q)$*, the norm defined by:*

$$|||\rho||| = \left(\|\rho\|^2_{L^2(Q)} + \frac{1}{2}\int_{\partial Q_+} (\,\tilde{u}\,|\,\tilde{n}\,)\,(T - t)\rho^2 d\tilde{\sigma}\right)^{1/2}$$

verifies

$$\|\rho\|_{L^2(Q)} \leq |||\rho||| \leq C\,|\rho|_{1,u}.$$

2. Existence and uniqueness results

A weak formulation

In $L^2(Q)$), a solution of equation $\partial_t c + \mathrm{div}\,(c\,u) = f$ corresponds to a minimizer in $\{\varphi \in H(u,Q);\ \gamma_{\widetilde{n}_-}(\varphi) = c_b;\ \gamma_{\widetilde{n}_+}(\varphi) = c_T\}$ of the following convex, $H(u,Q)$-coercive functional

$$J(c) = \frac{1}{2}\left(\int_Q \left(\widetilde{\mathrm{div}}(\widetilde{u}\,c) - f\right)^2 dx\,dt - \int_{\partial Q_-} c^2\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma}\right).$$

Its Gâteau derivative is

$$DJ(c)\varphi = \int_Q \left(\widetilde{\mathrm{div}}(\widetilde{u}\,c) - f\right)\,\widetilde{\mathrm{div}}(\widetilde{u}\,\varphi)\,dx\,dt - \int_{\partial Q_-} c\varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma}.$$

So a sufficient condition to get the least squares solution is following *weak formulation*

$$\int_{\boldsymbol{Q}} \widetilde{\mathbf{div}}(\widetilde{\boldsymbol{u}}\, \boldsymbol{c}) \cdot \widetilde{\mathbf{div}}(\widetilde{\boldsymbol{u}}\, \boldsymbol{\varphi})\, \boldsymbol{dx}\, \boldsymbol{dt} = \int_{\boldsymbol{Q}} \boldsymbol{f} \cdot \widetilde{\mathbf{div}}(\widetilde{\boldsymbol{u}}\, \boldsymbol{\varphi})\, \boldsymbol{dx}\, \boldsymbol{dt}, \quad (10)$$

$$\boldsymbol{\gamma}_{\widetilde{\boldsymbol{n}}_-}(\boldsymbol{c}) = \boldsymbol{c_b}, \quad (11)$$

$$\boldsymbol{\gamma}_{\widetilde{\boldsymbol{n}}_+}(\boldsymbol{c}) = \boldsymbol{c_T}, \quad (12)$$

for all $\boldsymbol{\varphi} \in \boldsymbol{H}_0(\boldsymbol{u}, \boldsymbol{Q})$.

Reduction to homogeneous Dirichlet problem

Let $C_b \in H(u, Q)$ be such that $\gamma_{\widetilde{n}_-}(C_b) = c_b$, and $C_T \in H(u, Q)$ be such that $\gamma_{\widetilde{n}_+}(C_T) = c_T$. Then $\rho = c - \big((1-t)C_b + tC_T\big)$ is the unique solution of

$$\int_Q \widetilde{\mathrm{div}}(\widetilde{u}\,\rho) \cdot \widetilde{\mathrm{div}}(\widetilde{u}\,\psi)\,dx\,dt = \int_Q \Big(f - \widetilde{\mathrm{div}}\,\big(\widetilde{u}\,\big((1-t)C_b + tC_T\big)\big)\Big) \cdot \widetilde{\mathrm{div}}(\widetilde{u}\,\psi)\,dx\,dt \quad (13)$$

for all $\psi \in H_0(u, Q)$. Moreover the solution of problem (13) is equivalent to the solution of (1). Therefore, modifying the source term when necessary, it is sufficient to only deal with homogeneous Dirichlet boundary conditions.

With the previous notations and hypothesis we have

Theorem 5. *Let* $u \in L^\infty(Q)^d$ *with* $\operatorname{div} u \in L^\infty(Q)$*, and assume that there exists* $C_b \in H(u,Q)$ *be such that* $\gamma_{\widetilde{n}_-}(C_b) = c_b$ *and there exists* $C_T \in H(u,Q)$ *be such that* $\gamma_{\widetilde{n}_+}(C_T) = c_T$*. Then the problem*

$$\int_Q \widetilde{\operatorname{div}}(\widetilde{u}\,\rho)\cdot\widetilde{\operatorname{div}}(\widetilde{u}\,\psi)\,dx\,dt = \int_Q \Big(f - \widetilde{\operatorname{div}}\left(\widetilde{u}\left((1-t)C_b + tC_T\right)\right)\Big)\cdot\widetilde{\operatorname{div}}(\widetilde{u}\,\psi)\,dx\,dt$$

has a unique solution. Moreover

$$\begin{aligned}|\rho|_{1,u} &= \left\|\widetilde{\operatorname{div}}(\widetilde{u}\,\rho)\right\|_{L^2(Q)} \\ &\le \|f\|_{L^2(Q)} + \left\|\widetilde{\operatorname{div}}\left(\widetilde{u}\left((1-t)C_b + tC_T\right)\right)\right\|_{L^2(Q)},\end{aligned}$$

and the function $c = \rho + \big((1-t)C_b + tC_T\big)$ *is the* space-time least squares solution *of (1)-(4).*

3. Numerical experiments

Finite Element formulation

Let $\{\varphi_1 \cdots \varphi_N\}$ be a basis of a finite element subspace $V_h \subset H_0(u, Q)$, (e.g. Q_1 Lagrange finite element). An approximation of problem (13) consists in finding $\rho_h \in V_h$ such that:

$$\int_Q \widetilde{\mathrm{div}}(\tilde{u}\,\rho_h) \cdot \widetilde{\mathrm{div}}(\tilde{u}\,\psi_h)\,dx\,dt = \int_Q \Big(f - \widetilde{\mathrm{div}}\,\big(\tilde{u}\,\Pi_h\big((1-t)C_b + tC_T\big)\big)\Big) \cdot \widetilde{\mathrm{div}}(\tilde{u}\,\psi_h)\,dx\,dt \quad (14)$$

for all $\psi_h \in V_h$, where $\rho_h = \sum_{i=1}^{N} \varphi_i(t,x)\rho_i$, and Π_h is the Lagrange interpolation operator.

Two examples

1. **The Hansbo's example revisited**
2. **Cardiac Images Interpolation**

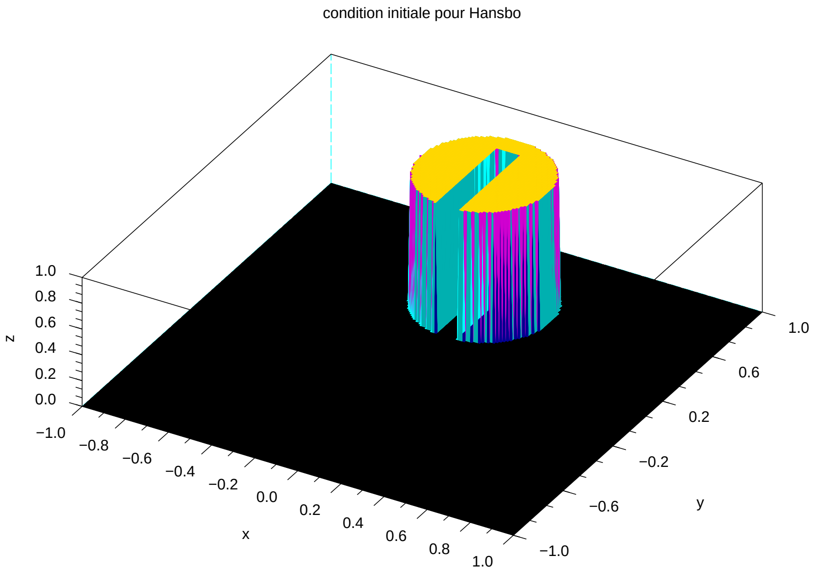
Hansbo example

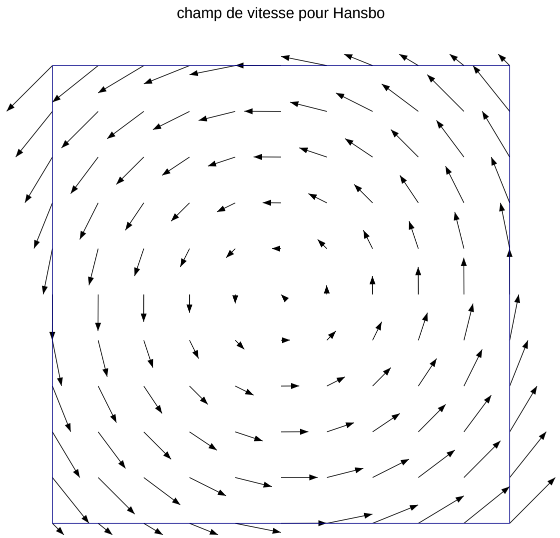
Transport equation benchmark proposed by Hansbo, for evaluating numerical schemes. $\Omega =]-1,1[^2$ and discretized in 100×100 Q_1-elements. The initial condition is

$$c(x,y,0) = \begin{cases} 1 & \text{if} \quad (|x| > 0.05 \text{ or } y > 0.7) \quad \text{and } R \leq 0.3 \\ 0 & \quad \text{elsewhere} \end{cases}$$

where $R = \sqrt{x^2 + (y-0.5)^2}$, the velocity field has the form $u(x,y,t) = (-y,x)$ and the final time is $T = 2\pi$.

Hansbo's example: initial condition and velocity field





The time interval $I = [0, 2\pi]$ is subdivided into 800 time steps.

- Initial and the final shape should be identical to the initial one.
- Then perturb the velocity field with a uniformly distributed noise. The final shape is distorted
- Finally use the previous method, specifying initial and final condition for the perturbed problem.

Movies for all situations

Comments on numerical methods

Problem with initial condition only

A time-marching method is considered, and a 100×100 regular grid is used. Then solve 800 times a $10'000$ unknown linear system. This can be performed on a usual small PC.
Total CPU time on this computer: 75 seconds

Problem with initial and final conditions

In this case a full $3D$ problem has to be solved. In this situation we get a $8'000'000$ unknown linear system

Cardiac images interpolation

Image interpolation: transformation of one image into an other. Widely used in motion of pictures.
Given a pair of images, find a sequence of intermediate images, such that the first image in the sequence is equal to the first given image (starting image) and the last image is equal to the second given image (ending image).

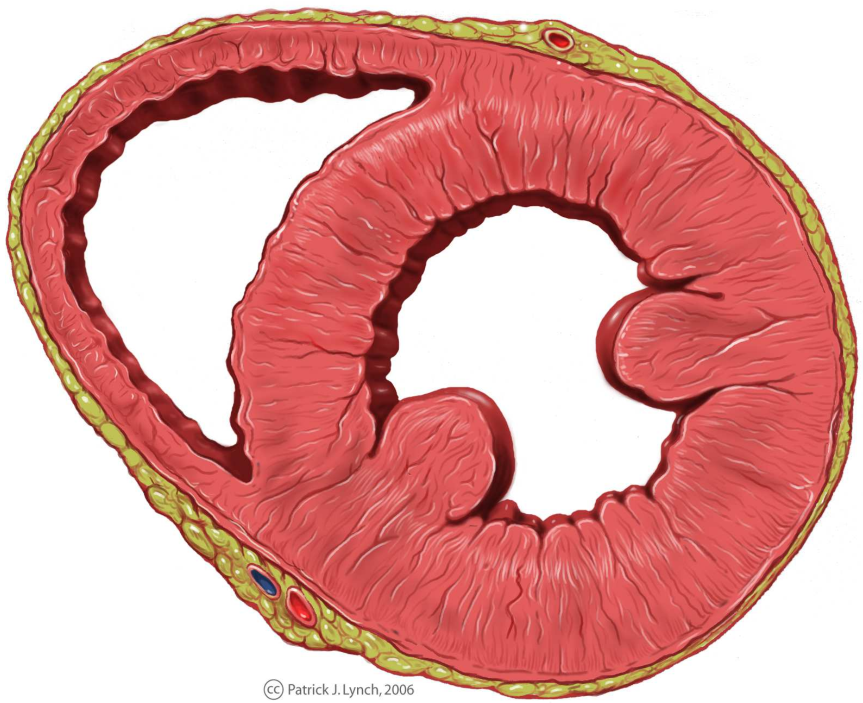
In cardiac Magnetic Resonance Imaging (MRI), several cross-section images acquired to map the heart in 3D and at a collection of discrete time samples over the cardiac cycle.
Then, extrapolate from these input data the complete cardiac cycle. Prerequisite: velocity field between image slices.
Two main approaches commonly used for estimating the velocity fields.

- The tracking landmark technique

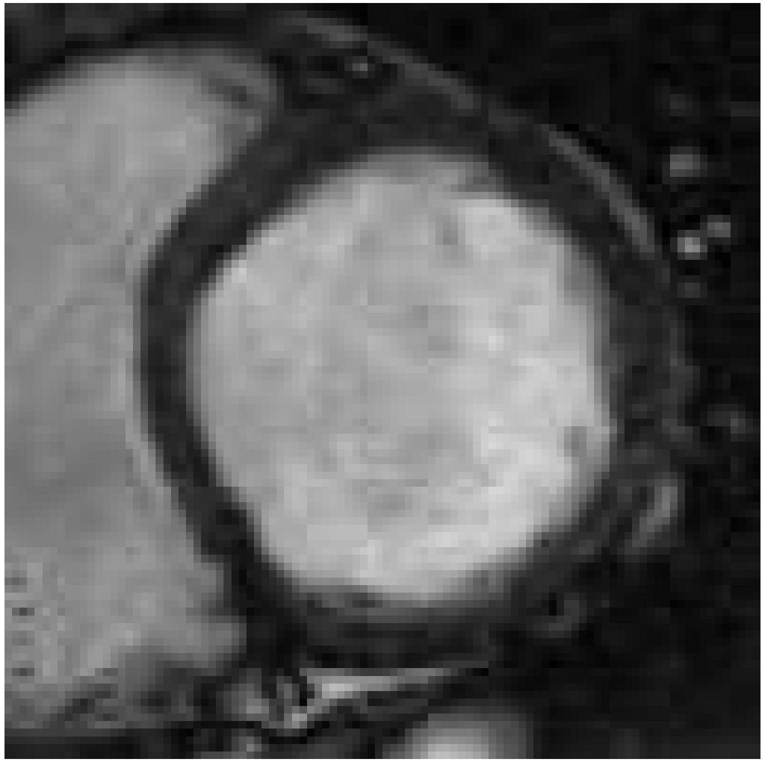
- Optimization problem, using optical flow equation and regularization.

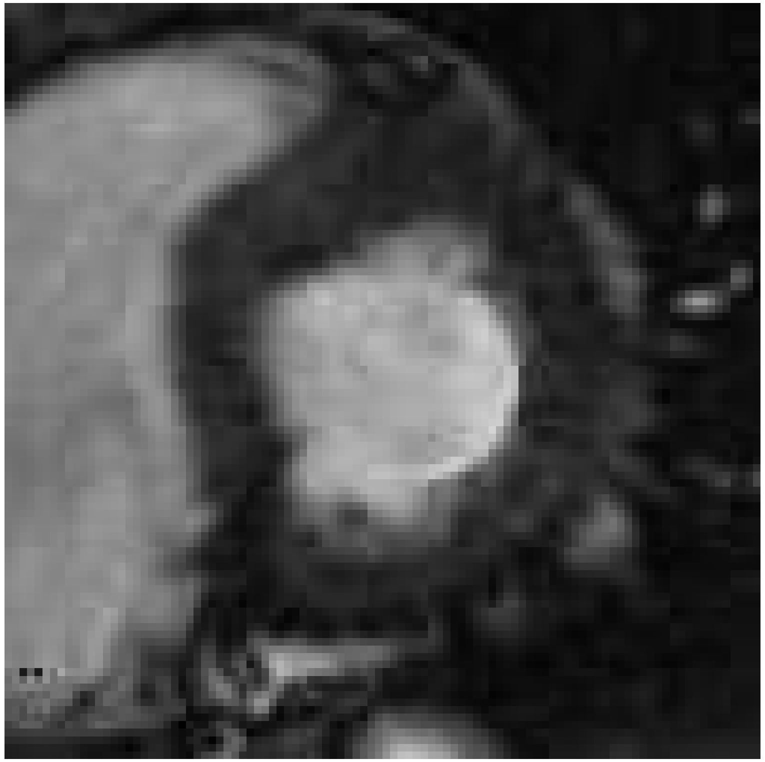
Velocity field is computed from the end of diastole slice of a left ventricle of a heart to the end of systole slice.

Left ventricle on the right, right ventricle on the left



End of diastole, and end of systole





Using the same methods as before, the reconstruction of a systole is presented. The total time between the end of the diastole and the end of the systole in the present case is $T_s = 7ms$. The time interval $I = [0, T_s]$ is subdivided into 100 time steps. The images are defined as a 100×100 grid of grey levels.

Different situations are presented.

- The motion of the left ventricle during the systole, using a time varying velocity field reconstructed form optimization methods (J. Ehrhardt, D. Saring, and H. Handels, 2007).
- The same situation, but with a mean velocity field in time.
- Initial and final conditions are imposed, and a mean velocity field in time is used.

Movies for all situations

Thank you