Mathematical Problems Arising in Electrostatic MEMS
or
On some nonlinear eigenvalue problems

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Micro-Electro-Mechanical Systems?

They are essentially electrostatically actuated micro-plates which lie at the roots of micro-system technology.

We consider here a model for an electrostatic-controlled tunable capacitor that is widely used in:

- micro-resonators,
- optical micro-switches,
- chemical sensors, micro-mirrors,
- accelerometers for airbag development of automobiles,
- micro-pumps for ink jet printer heads,
- micro-valves,
- shuffle motors,
- micro- and nano-tweezers,
- among many other devices....

MEMS references

- Pelesko & Bernstein, *Modeling MEMS and NEMS*, Chapman & Hall/CRC, Boca Raton, FL, 2003
- Bernstein, Guidotti & Pelesko, Proc. of Modeling and Simulation of Microsystems ’00.
- Chan & Dutton, Proceedings of SPIE ’99.
- Flores, Mercado & Pelesko, Proceedings of ICMENS ’03.
- Pelesko, Bernstein & Mc Cuan, Proceedings of MSM ’03.
- Seeger & Crary, Proc. of 1997 International Conference on Solid-State Sensors and Actuators ’97.
- Michael Ward and Alan Lindsay, 2004-09
- (Another) Guo and J-C Wei, 2006-09
- F.H. Lin and Yang 2007
- Castorina, Esposito & Sciunzi, CPAA to appear
- Dancer, AIHP ’08
- Farina, J. Math. Pures Appl. ’07

However, not widely accepted models (Naifeh, etc...)

Basic MEMS Model

- Lower part consists of a thin and deformable microplate that is held fixed along its boundary $\partial\Omega$, where Ω is a bounded domain in $\mathbf{R}^2$.
- The dielectric elastic micro-plate lies below a parallel rigid grounded plate, say at level $z = 1$. It is coated with a negligibly thin metallic conducting film.
- When a voltage V is applied to the conducting film, it deflects towards the upper plate.
- If the applied voltage V is increased beyond a certain critical value V^*, then the mechanical forces defined by the deformable plate can no longer resist the opposing electrostatic force, thereby leading it to touch the grounded plate.

The dynamic deflection $u = u(x,t)$ of the membrane on a bounded domain Ω in $\mathbb{R}^2$ satisfies the evolution equation

$$\begin{aligned}
\rho A \frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} &= T\Delta u - B\Delta^2 u - \frac{C^2V^2}{(1-u)^2} \quad \text{on} \quad \Omega\times\mathbf{R}^+,\\
u(x,t) &= B\frac{\partial u}{\partial \eta}(x,t) = 0 \quad \text{on} \quad \partial\Omega\times\mathbf{R}^+,\\
u(x,0) &= A\frac{\partial u}{\partial t}(x,0) = 0 \quad \text{on} \quad \Omega.
\end{aligned}$$

- ρ is mass density per unit volume and A is its thickness;
- a is a damping intensity;
- T is the tension constant in the stretching energy;
- B for bending constant and is given by $B = \frac{2A^3Y}{3(1-\nu^2)}$ where Y is the Young modulus and ν is the Poisson ratio;
- Membrane and top plate subjected to a capacitance C and an electric voltage V;
- Undeflected membrane as initial condition.
- Membrane clamped and fixed at boundary.

Stationary equations under study

The stationary state is described by the following fourth-order equation describing the deflection $u = u(x)$ of a circular membrane with a clamped boundary condition

$$(IV)_B \qquad \begin{cases} B\Delta^2 u - T\Delta u = \dfrac{\lambda}{(1-u)^2} & \text{in } B, \\ 0 \le u < 1 & \text{in } B, \\ u = \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial B, \end{cases}$$

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If membrane is assumed to be perfectly elastic and has no rigidity, then second order nonlinear eigenvalue problem on a general bounded domain Ω:

$$(II)_{\Omega,f} \qquad \begin{cases} -\Delta u = \dfrac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega, \\ 0 \le u < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Dynamic case under study The dynamic deflection $u = u(x,t)$ will be considered after we make a huge simplification to the model by ignoring rotational inertial which would have given the equation a hyperbolic character.
In other words, we assume that the membrane thickness A is negligible to end up with the following parabolic equation on a time interval $(0,T)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \dfrac{\lambda f(x)}{(1-u)^2} & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = 0 & x \in \Omega. \end{cases}$$

Pull-In Voltage and Steady-States

Pull-In Voltage λ^* is:

$$\lambda^* = \sup\{\lambda > 0 \mid (II)_{\lambda,\Omega,f} \quad \text{resp., } (IV)_\lambda \text{ has at least one solution }\}.$$

Theorem: There exists $0 < \lambda^* < +\infty$ such that

- $\lambda < \lambda^* \Longrightarrow$ at least one solution;
- $\lambda > \lambda^* \Longrightarrow$ no solution

Issues

- λ is small: a stable deflection, and when $\lambda > \lambda^*$ (pull-in voltage): snap-through (touchdown) “Pull-In" Instability, which limits the stable operation regime, *e.g.* Micropumps and microresonators.
- What happens when $\lambda = \lambda^*$?
- Estimate $\lambda^*(\Omega, f)$ in terms of the geometry of the domain and the permittivity profile f.

Lower Bounds for λ^*

Isoperimetry:

$$\lambda^*(\Omega, f) \geq \lambda^*(B_R, f^*), \text{ where } |B_R| = |\Omega|.$$

Noticed numerically (Guo-Pan-Ward (2004))
Can be established mathematically (Ghoussoub-Guo 2005).

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Theorem: *For the model* $(II)_{\Omega,f}$,

$$\lambda^*(\Omega, f) \geq \max\left\{\frac{8N}{27}, \frac{6N-8}{9}\right\} \frac{1}{\sup_{x\in\Omega} f} \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}}.$$

Upper Bounds for λ^*:

$$\lambda^*(\Omega, f) \leq \min\left\{\bar{\lambda}_1 := \frac{4\mu_\Omega}{27 \inf_{x\in\Omega} f(x)},\ \bar{\lambda}_2 := \frac{\mu_\Omega}{3\int_\Omega f\phi_\Omega\, dx}\right\}.$$

$(\mu_\Omega, \phi_\Omega)$ being the first eigenpair of the Laplacian on $H_0^1(\Omega)$.

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- Pelesko for $\bar\lambda_1$
- Guo-Pan-Ward for $\bar\lambda_2$.
- Ghoussoub-Guo: If Ω is star shaped with respect to 0 (i.e., $x \cdot \nu \ge a > 0$ for all $x \in \partial\Omega$). Then

$$\lambda^*(\Omega, 1) \le \frac{(N+2)^2|\partial\Omega|}{8aN|\Omega|},$$

where $|\Omega|$ is the volume of Ω and where $|\partial\Omega|$ denotes the surface area of the boundary.

Analytic Bounds for λ^* for circular membranes

Suppose B is the unit ball in $\mathbf{R}^N$, and $f(x) = |x|^\alpha$.

1. For $\alpha >> 1$,

$$\lambda^*(B, |x|^\alpha) \approx 0.789\left(1 + \frac{\alpha}{2}\right)^2.$$

2. If B is the unit ball in $\mathbf{R}^2$. Then, for all $\alpha > -2$.

$$\lambda^*(B, |x|^\alpha) = 0.789\left(1 + \frac{\alpha}{2}\right)^2,$$

3. If $N \geq 8$ and $0 \leq \alpha \leq \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. Then

$$\lambda^*(B, |x|^\alpha) = \frac{(2+\alpha)(3N+\alpha-4)}{9}.$$

Minimal and Stable Steady-States:

Definition: *$u_\lambda(x)$ is a minimal solution, if for any other solution $u(x)$ of $(II)_\lambda$ (or $(IV)_\lambda$) we have $u_\lambda(x) \leq u(x)$ on Ω.*

Set $\quad L_{u,\lambda} = -\Delta - \frac{2\lambda f(x)}{(1-u)^3} \quad$ resp., $\quad L_{u,\lambda} = \beta\Delta^2 - \Delta - \frac{2\lambda f(x)}{(1-u)^3}$

Definition: u_λ is a stable solution, if $L_{\cdot,\lambda}$ is positive on $H_0^1(\Omega)$ (resp., $H_0^2(\Omega)$).

Theorem: Minimal Solution $\Longleftrightarrow$ Stable Solution.

Morse Index $m(u,\lambda) := \sharp\{$ negative eigenvalues of $L_{u,\lambda}\}$.

For unstable (i.e., non-minimal) solution $m(u,\lambda)$ is the dimension of the negative space of $L_{u,\lambda}$.

Pull-in Distance:

Theorem: If $0 \leq \lambda < \lambda^*$, there exists a unique minimal (i.e, stable) u_λ of $(S)_\lambda$, and the map $\lambda \to u_\lambda$ is strictly increasing.

Define $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$, the extremal solution of $(S)_\lambda$.

u^* is singular if $\|u_{\lambda^*}\|_\infty = 1$

u^* is regular if $\|u_{\lambda^*}\|_\infty < 1$.

Pull-in Distance is defined as $P(\Omega, f, N) = \|u_{\lambda^*}\|_\infty$.

Estimates on the Pull-in Distance:

Theorem: (Cowan-Ghoussoub) Suppose Ω is a bounded domain in $\mathbf{R}^N$ and $0 \leq f(x)$ Holder continuous. Then

$$P(\Omega, f) \geq 1 - \min\left\{\frac{2}{3}\left(\frac{\sup_\Omega f}{\inf_\Omega f}\right)^{\frac{1}{3}}, \left(\frac{2\sup_\Omega f}{3\int_\Omega f\phi_\Omega dx}\right)^{\frac{1}{3}}\right\}.$$

In particular if $f(x) \equiv 1$ then $P(\Omega, 1) \geq \frac{1}{3}$.
Observed numerically by Pelesko (Conventional wisdom!).
On the other hand, if B is the unit ball in $\mathbf{R}^N$, then

1. For $N = 1$. Then $P(B, 1) \leq .49...$
2. For $N = 2$. Then $P(B, 1) \leq .55...$
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4. If $N = 2$, then $P(B, |x|^\alpha) = P(B, 1)$ for any α. (Boundary layer appears as α increases). Observed numerically by Guo-Pan-Ward.

Second order equation: Solution set

- If $N = 1$, $f \equiv 1$, $\Omega = B$:
Exactly 2 solutions for $\lambda \in (0, \lambda^*)$.

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- If $2 \leq N \leq 7$, $f(x) = g(x)|x|^\alpha$ with $g \geq c > 0$ and any Ω:
  - uniqueness for small λ if Ω star-shaped (Esposito-Gh. 08);
  - regularity of extremal solution (Gh-Guo 08);
  - exactly 2 solutions close to λ^* and
  - a curve $(\lambda(t), u(t))_{t \geq 0}$ of solutions starting from $(0, 0)$ and $\|u(t)\|_\infty \to 1$ as $t \to +\infty$, with infinitely many bifurcation or turning points (Esposito-Gh. 08);

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- If $N \geq 8$, $f \equiv 1$, $\Omega = B$:
Extremal solution is singular. Just the minimal branch?
(Gh-Guo 05);

Fourth order equation: $f(x) \equiv 1$ and $\Omega = B_1(0) \subset \mathbb{R}^N$

Cowan-Esposito-Ghoussoub-Moradifam 2009

- If $N = 1, 2$, at least 2 solutions for any $\lambda \in (0, \lambda^*)$.
- If $3 \leq N \leq 8$, the extremal solution is regular hence at least one bifurcation.
- If $N \geq 9$, the extremal solution is singular.

$N = 8$ is the critical dimension for 4th order equation, while
$N = 7$ is the critical dimension for 2d order equation.

Fourth order much more intricate. Involves new Hardy-Rellich inequalities.

Compactness along Higher Branches

$$(II)_{\Omega,f} \qquad \begin{cases} -\Delta u = \dfrac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega, \\ 0 \le u < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem: If Ω is a bounded domain in $\mathbf{R}^N$ with $2 \le N \le 7$ and $f(x) = g(x)|x|^\alpha$ with $g \ge c > 0$. Let $\{\lambda_n\}_{n \in N}$ be a sequence such that $\lambda_n \to \lambda \in [0, \lambda^*]$ and let u_n be an associated solution of $(II)_{\lambda_n}$. Then the following are equivalent:

1. $\max\limits_{\Omega} u_n \to 1$ as $n \to +\infty$;
2. $\displaystyle\int_\Omega \left(\frac{f(x)}{(1-u_n)^3} \right)^{\frac{N}{2}} \to +\infty$ as $n \to +\infty$;
3. $m(u_n, \lambda_n) \to +\infty$ as $n \to +\infty$.

Blow-up Analysis

Suppose $\lambda_n \to \lambda \in [0, \lambda^*]$ and $u_{\lambda_n}(x) \to 1$ at some $x_0 \in \bar{\Omega}$. Set

$$\varepsilon_n = 1 - u_{\lambda_n}(x_0) \to 0, \quad U_n(y) = \frac{1 - u_{\lambda_n}(\varepsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} y + x_0)}{\varepsilon_n}.$$

Then $\{U_n\}_n$ converges to a solution of

$$\begin{cases} \Delta U = \dfrac{|y|^\alpha}{U^2} & \text{in } \mathbb{R}^N, \\ U(y) \geq C > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{L}$$

What we can know about (L) would tell us more about the branches of $(II)_\lambda$!

Limiting Problem:

Theorem: Assume either

- $1 \leq N \leq 7$
- or $N \geq 8$ with $\alpha > \alpha^{**}(N) := \frac{4-6N+3\sqrt{6}(N-2)}{4}$

If U is a solution of

$$\begin{cases} \Delta U = \dfrac{|y|^\alpha}{U^2} & \text{in } \mathbb{R}^N, \\ U(y) \geq C > 0 & \text{in } \mathbb{R}^N. \end{cases} \qquad (L)$$

Then,

1. U is necessarily unstable (i.e at least

$$\mu_1(U) = \inf\left\{ \int_{\mathbb{R}^N} \left(|\nabla\phi|^2 - \frac{2|y|^\alpha}{U^3}\phi^2 \right); \phi \in C_0^\infty(\mathbb{R}^N), \int_{\mathbb{R}^N} \phi^2 = 1 \right\} < 0.$$

2. If $N \geq 8$ and $0 \leq \alpha \leq \alpha^{**}(N)$, then (L) has at least one solution U which is semi-stable.

Restoring compactness for $N \geq 8$ with $f(x) = |x|^\alpha$

Theorem: $\Omega = B_1(0)$, $N \geq 8$ and $\alpha^{**}(N) := \frac{4-6N+3\sqrt{6}(N-2)}{4}$, then

1. If $0 \leq \alpha \leq \alpha^{**}(N)$, then

$$u^*(x) = \lim_{\lambda \uparrow \lambda^*} u_\lambda(x) = 1 - |x|^{\frac{2+\alpha}{3}}, \quad \lambda^* = \frac{(2+\alpha)(3N+\alpha-4)}{9}.$$

2. If $\alpha > \alpha^{**}(N)$, then $(II)_{\lambda,|x|^\alpha}$ restores all regularity of the case where $1 \leq N \leq 7$ and $f(x) \equiv 1$, including the compactness of u_λ as λ close to λ^*, i.e., no more critical dimension!

Bifurcation Diagrams $\Longrightarrow$

Dynamic Deflection:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \dfrac{\lambda f(x)}{(1-u)^2} & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0 = 0 & x \in \Omega. \end{cases}$$

Theorem (Gh-Guo 2005, Same pull-in voltage $\lambda^*(\Omega, f)$)

1. If $\lambda < \lambda^*$, then u globally converges to the minimal steady-state u_λ.
2. If $\lambda > \lambda^*$, then u must touchdown at finite time T_λ.
3. If $\lambda = \lambda^*$, then u globally converges to the extremal solution $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$.
   (i.e., if u^* is singular then u touches down at infinite time).

Define the touchdown (i.e. quenching) time

$$T_\lambda(\Omega, f, u) = \sup\left\{ t \in (0, +\infty);\ \sup_{x \in \Omega} u(x,t) < 1 \right\}.$$

which is the maximal time of existence for $u(x,t)$.

Estimates of Touchdown Time T_λ: $\lambda > \lambda^*$

Recall $\bar{\lambda}_1 := \frac{4\mu_\Omega}{27 \inf_{x\in\Omega} f(x)} (> \lambda^*)$ and $\bar{\lambda}_2 := \frac{\mu_\Omega}{3\int_\Omega f\phi_\Omega\, dx} (> \lambda^*)$.

Theorem: (Guo-Pan-Ward 2004)

1. If $\lambda > \bar{\lambda}_1$, then $T_\lambda \le T_1 := \int_{-1}^{0} \Big[\mu_\Omega s + \frac{\lambda \inf_{x\in\Omega} f(x)}{(1+s)^2}\Big]^{-1} ds.$
2. If $\lambda > \bar{\lambda}_2$, then $T_\lambda \le T_2 := -\frac{1}{\mu_\Omega} \log\Big[1 - \frac{\mu_\Omega}{3\lambda}\Big(\int_\Omega f\phi_\Omega\, dx\Big)^{-1}\Big].$

Theorem: (Gh-Guo, 2006 and 2008) If $\lambda > \lambda^*$, then

$$T_\lambda \le T_3 := \frac{8(\lambda+\lambda^*)^2}{3\inf_\Omega f(\lambda-\lambda^*)^2(\lambda+3\lambda^*)}\Big[1+\Big(\frac{\lambda+3\lambda^*}{2\lambda+2\lambda^*}\Big)^{1/2}\Big].$$

If also $\int_\Omega \frac{\phi^*(x)}{f(x)} dx < \infty$, there exists $C = C(f,\Omega)$ with

$$T_\lambda \le T_4 := C\cdot\Big(\lambda-\lambda^*\Big)^{-\frac{1}{2}}.$$

Here $\phi^* > 0$, $\int_\Omega \phi^*(x)dx = 1$ and $-\Delta\phi^* - \frac{2\lambda^*\phi^* f(x)}{(1-u^*)^3} = 0$ in Ω. .

Singularity of extremal solution for $(II)_B$ when $N \geq 8$

Lemma: If u is an $H_0^1(\Omega)$-weak solution of (II_λ) that is singular (i.e., $\| u \|_\infty = 1$) and semi-stable, i.e.,

$$\int_\Omega |\nabla \phi|^2 \geq \int_\Omega \frac{2\lambda}{(1-u)^3}\phi^2 \quad \forall \phi \in H_0^1(\Omega),$$

then $\lambda = \lambda^*$ and $u = u^*$..

If now Ω is the unit ball of $\mathbf{R}^N$, then $u(x) = 1 - |x|^{\frac{2}{3}}$ is a singular $H_0^1(B)$-weak solution of (II_λ) that corresponds to the voltage

$$\lambda(N) = \frac{2(3N-4)}{9}.$$

It is the extremal solution when it is semi-stable, i.e., when

$$\int_B |\nabla \phi|^2 \geq \int_{B_1} \frac{2\lambda}{(1-u)^3}\phi^2 = 2\lambda(N)\int_B \frac{\phi^2}{|x|^2}.$$

But this is implied by Hardy's inequality whenever

$$2\lambda(N) \leq \frac{(N-2)^2}{4} \text{ or, equivalently, if } N \geq 8.$$

Singularity of extremal solution for $(IV)_B$ when $N \geq 9$

Same game with $u(x) = 1 - |x|^{\frac{4}{3}}$: a singular $H_0^2(B)$-weak solution of (IV_λ) that corresponds to the voltage

$$\lambda(N) := \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3}).$$

It is the extremal solution when it is semi-stable, i.e., when

$$\int_{B_1} |\Delta\phi|^2 \geq \int_{B_1} \frac{2\lambda}{(1-u)^3}\phi^2 = 2\lambda(N) \int_{B_1} \frac{\phi^2}{|x|^4}.$$

But this is implied by the Hardy-Rellich inequality whenever

$$2\lambda(N) \leq \frac{N^2(N-4)^2}{16} \text{ or, equivalently, if } N \geq 9.$$

Problem: $u(x) = 1 - |x|^{\frac{4}{3}}$ does not satisfy the boundary condition!!!!.

So try

$$u_m(x) := 1 - \frac{3m}{3m-4}|x|^{4/3} + \frac{4}{3m-4}|x|^m.$$

It works for $N \geq 17$, by using an improved Hardy-Rellich inequality. for all $\phi \in H_0^2(B)$

$$\int_B (\Delta\phi)^2\, dx \geq \frac{N^2(N-4)^2}{16} \int_B \frac{\phi^2}{|x|^4}\, dx + C \int_B \phi^2\, dx.$$

What happend for $9 \leq N \leq 16$?

Enter Moradifam (2009)

To establish semi-stability for u_3 and $u_{2.8}$, one needs to establish the following improved Hardy-Rellich inequalities:

- **Case $10 \le N \le 16$:** we need that for all $\phi \in H_0^2(B)$

$$\int_B (\Delta\phi)^2\,dx \;\ge\; \frac{(N-2)^2(N-4)^2}{16}\int_B \frac{\phi^2\,dx}{(|x|^2-|x|^{\frac{N}{2}+1})(|x|^2-|x|^{\frac{N}{2}})} + \frac{(N-1)(N-4)^2}{4}\int_B \frac{\phi^2\,dx}{|x|^2(|x|^2-|x|^{\frac{N}{2}})}.$$

- **Case $N = 9$:** we need that for all $\phi \in H_0^2(B)$

$$\int_B (\Delta\phi)^2\,dx \ge \int_B Q(|x|)\left(P(|x|) + \frac{N-1}{|x|^2}\right)\phi^2\,dx,$$

where $P(r) = \frac{p''(r)+\frac{(N-1)}{r}p'(r)}{p}$ and $Q(r) = \frac{q''(r)+\frac{(N-3)}{r}q'(r)}{q}$,

$p(r) := r^{-\frac{N}{2}+1} + r - 1.9$; $q(r) := r^{-\frac{N}{2}+2} + 20r^{-1.69} + 10r^{-1} + 10r + 7r^2 - 48.$

Amazing case of SERENDIPITY!!

General Hardy Inequalities (Ghoussoub-Moradifam 2007)

Let V, W be positive radial C^1-functions on $(0, R)$ such that $\int_0^R \frac{1}{r^{n-1}V(r)}dr = +\infty$. Are then equivalent:

1. (V, W) is a Bessel pair on $(0, R)$, i.e.,

$$y''(r) + (\frac{n-1}{r} + \frac{V_r(r)}{V(r)})y'(r) + \frac{W(r)}{V(r)}y(r) = 0$$

   has a positive solution on the interval $(0, R)$.
2. $\int_B V(x)|\nabla u|^2 dx \geq \int_B W(x)u^2 dx$ for all $u \in C_0^\infty(B)$.
3. If $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < n-2$, then the above are equivalent to: for all radial $u \in C_{0,r}^\infty(B)$

$$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B (\frac{V(x)}{|x|^2} - \frac{V_r(|x|)}{|x|})|\nabla u|^2 dx.$$